Caringbah High School

2014

Trial HSC Examination

Mathematics Extension 1

General Instructions

- Reading time – 5 minutes
- Working time – 2 hours
- Write using black or blue pen (Black pen is preferred)

- Board-approved calculators may be used
- A table of standard integrals is provided at the back of this paper

- In Questions 11–14, show relevant mathematical reasoning and/or calculations

Total marks – 70

Section I Pages 2 – 4

10 marks

- Attempt Questions 1–10
- Allow about 15 minutes for this section

Section II Pages 5 – 8

60 marks

- Attempt Questions 11–14
- Allow about 1 hour and 45 minutes for this section

Question 1 - 10 (1 mark each) Answer on page provided.

1) The remainder when $P(x) = 2x^3 + x^2 - 5x - 3$ is divided by $x + 1$ is:

A) -3 B) -5 C) 1 D) -1

2) $\int \frac{1}{3 + x^2}\,dx$ is given by:

A) $\tan^{-1}\sqrt{3}x + c$

B) $\frac{1}{3}\tan^{-1}\frac{x}{3} + c$

C) $\frac{1}{3}\tan^{-1}3x + c$

D) $\frac{1}{\sqrt{3}}\tan^{-1}\frac{x}{\sqrt{3}} + c$

3) Given the function $f(x):\ y = \frac{2}{x + 1}$, then its inverse function $f^{-1}(x)$ in terms of x is given by:

A) $y = \frac{2 - x}{x}$

B) $y = \frac{x + 1}{2}$

C) $y = \frac{2 - x}{2}$

D) $y = \frac{2 + x}{x}$

4) $\lim_{x \to 0} \frac{\sin 3x}{x}$ is equal to:

A) 1 B) $\frac{1}{3}$ C) -3 D) 3

5) The solution to the equation $3^{x-1} = 5$ is given by:

A) $x = \dfrac{\ln 5}{\ln 3} + 1$

B) $x = \dfrac{\ln 5}{\ln 3} - 1$

C) $x = \dfrac{\ln 3}{\ln 5} + 1$

D) $x = \dfrac{\ln 3}{\ln 5} - 1$

6) The general solution to the equation $2\sin\theta = \sqrt{3}$ is given by:

A) $\theta = k\pi + \dfrac{\pi}{3}$

B) $\theta = k\pi + (-1)^k \dfrac{\pi}{3}$

C) $\theta = k\pi + (-1)^k \dfrac{\pi}{6}$

D) $\theta = 2k\pi \pm \dfrac{\pi}{3}$

7) If two of the roots of $2x^3 - gx^2 + hx - 8 = 0$ are -1 and 2, the other root is:

A) -2 B) 2 C) -4 D) 4

8) If $f(x) = \sin^2(3-x)$, then $f'(0)$ is equal to:

A) $-2\cos(3)$

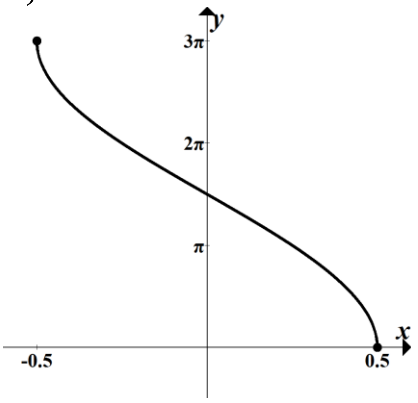
B) $-2\sin(3)\cos(3)$

C) $2\sin(3)\cos(3)$

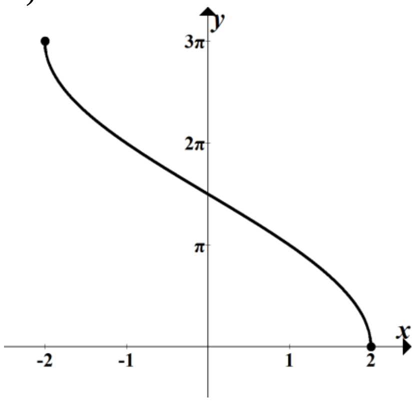
D) $6\sin(3)\cos(3)$

9) The graph of $y = 3\cos^{-1}\left(\frac{x}{2}\right)$ is:

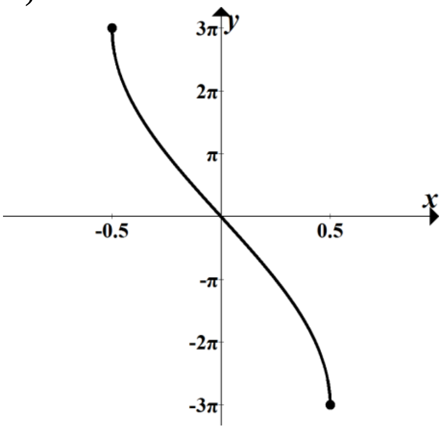
A)



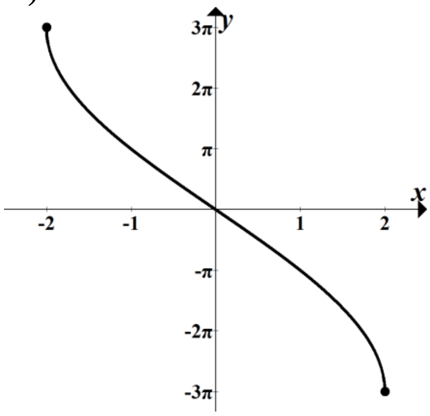
B)



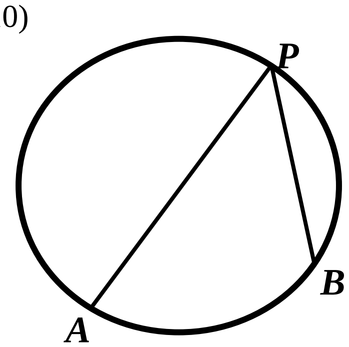
C)



D)



10)



AP is a diameter of the circle.

If $\angle APB = 40°$, then $\angle PAB$ is:

A) 60°

B) 40°

C) 50°

D) not enough information

END OF MULTIPLE CHOICE QUESTIONS

Section II

60 marks
Attempt all questions 11−14
Allow about 1 hour and 45 minutes for this section

Answer each question in a SEPARATE writing booklet. Extra writing booklets are available.

In Questions 11−14, your responses should include relevant mathematical reasoning and/or calculations.

Question 11 (15 marks) Start a NEW booklet. **Marks**

a) Solve $\frac{2x}{x-1} < 3$. 2

b) Find the acute angle between the lines $x - 2y + 3 = 0$ and $3x + y + 6 = 0$. 2

[Answer to the **nearest degree**]

c) Use the substitution $t = \tan\frac{\theta}{2}$ to express $1 + \tan\theta\tan\frac{\theta}{2}$ as a fraction 2

in simplest form.

d) Find $\int \sin^2 x \, dx$. 2

e) Given that a root of $y = 3x + \ln x - 1$ lies close to $x = 0.4$, use Newton's method 2

once to find an improved value of that root correct to 2 decimal places.

f) Solve $\cos 2\theta = \cos\theta$ where $0 \le \theta \le 2\pi$. 2

g) i) Write down the expansion of $\sin(A - B)$. 1

ii) Hence show that $\sin 15° = \frac{\sqrt{6} - \sqrt{2}}{4}$. 2

Question 12 (15 marks) Start a NEW booklet. **Marks**

a) The variable point $\left(3t, 4t^2\right)$ lies on a parabola. 2

Find the cartesian equation of this parabola.

b) Find the domain of $y = \ln\left(\sin^{-1} x\right)$. 2

c) Consider the function $f\left(x\right) = \dfrac{3x}{x^2 - 1}$.

i) Show that the function is odd. 1

ii) Show that the function is decreasing for all values of x. 2

iii) Neatly sketch the graph of $f\left(x\right) = \dfrac{3x}{x^2 - 1}$ showing clearly the equations of any asymptotes. 2

d) Use the substitution $u = x^2 - 2$ to evaluate $\displaystyle\int_{\sqrt{3}}^{\sqrt{11}} \frac{x}{\sqrt{x^2 - 2}}\, dx$ 3

e) At time t hours after an oil spill occurs, a circular oil slick has a radius of 'r' kilometres, where $r = \sqrt{t + 1} - 1$. 3

Find the rate at which the area of the slick is increasing when the radius is 1 kilometre.

Question 13 (15 marks) Start a NEW booklet. **Marks**

a) Find the exact value of $\cos\left(\sin^{-1}\left(\frac{3}{4}\right)\right)$. 1

b) i) Given that $A > 0$ and $0 < \alpha < \frac{\pi}{2}$, show that when $\cos x - \sin x$ is expressed in the form $A\cos(x + \alpha)$, that $A = \sqrt{2}$ and $\alpha = \frac{\pi}{4}$. 2

ii) Hence solve $\cos x - \sin x = \frac{1}{\sqrt{2}}$ for $0 \leq x \leq 2\pi$. 2

c) A cake is cooling after being taken out of the oven. The surrounding temperature in the room is $20°C$.

At a time 't' minutes, its temperature T decreases according to the equation $\frac{dT}{dt} = -k(T - 20)$ where 'k' is a positive constant. The initial temperature of the cake immediately after removal from the oven was $180°C$ and it cools to $120°C$ after 5 minutes.

i) Show that $T = 20 + Ae^{-kt}$ is a solution to $\frac{dT}{dt} = -k(T - 20)$, where A is a constant. 1

ii) Find the values of A and k. 2

iii) How long will it take for the cake to cool to $60°C$? 2
(Answer to the nearest minute)

Question 13 continues on the next page

Question 13 continued. **Marks**

d) The two points $P\left(2ap, ap^2\right)$ and $Q\left(2aq, aq^2\right)$ are on the parabola $x^2 = 4ay$.

i) Show that the equation of the tangent at P is given by $y = px - ap^2$. 1

ii) The tangents at P and Q meet at T. 2

Show that T has coordinates $\left[a\left(p+q\right), apq\right]$.

iii) P and Q move in such a way that $\angle POQ$ is always 90°. (O is the origin). 1

Show that $pq = -4$.

iv) Hence deduce the locus of T. 1

End of Question 13.

Question 14 (15 marks) Start a NEW booklet. **Marks**

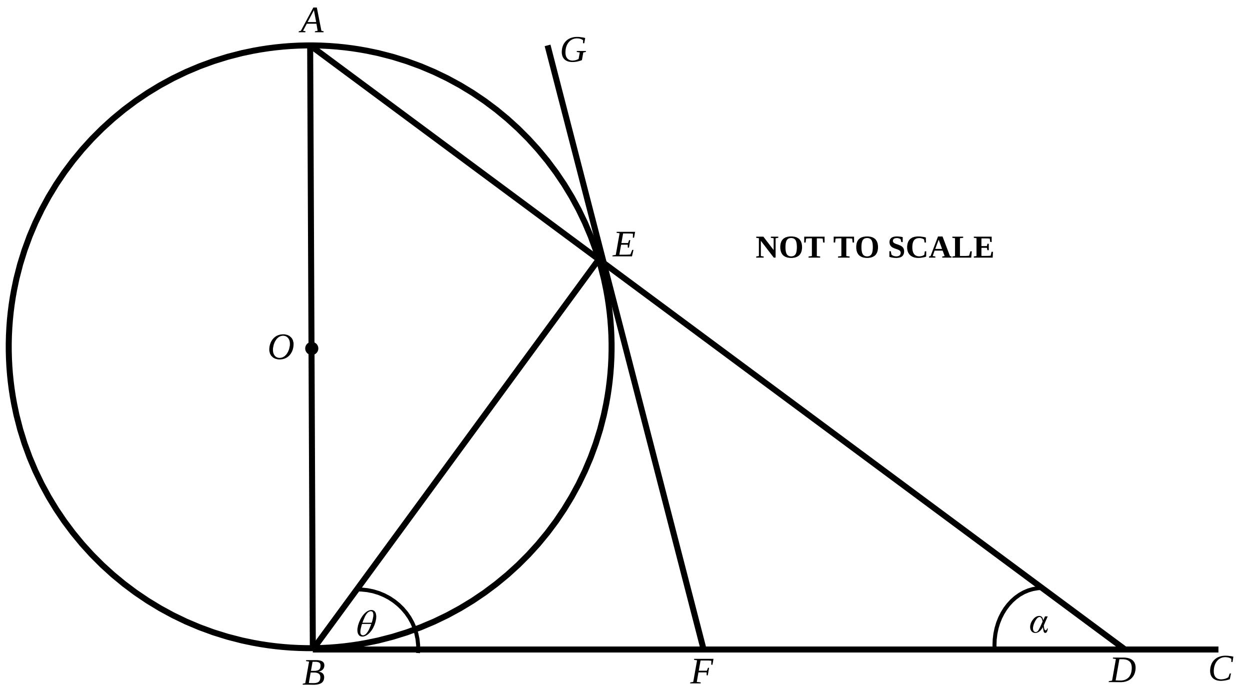
a) The polynomial $P(x) = 4x^3 + kx + 6$ has $(x+3)$ as a factor. 2

Find the value of k and express $P(x)$ in the form $(x+3)Q(x)$.

b) In the diagram below, *AB* is the diameter of the circle centre *O*, and *BC* is tangential to the circle at *B*.

The line *AD* intersects the circle at *E* and *BC* at *D*. The tangent to the circle at *E* intersects *BC* at *F*.

Let $\angle EBF = \theta$ and $\angle EDF = \alpha$.



Answer this question on the page provided

i) Prove that $\angle FED = \dfrac{\pi}{2} - \theta$ 2

ii) Prove that $BF = FD$. 2

Question 14 continues on the next page

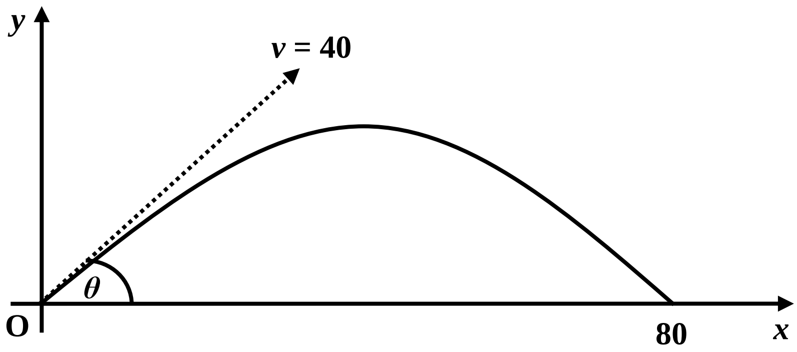
Question 14 continued. **Marks**

c) Prove by Mathematical Induction that 3

$$1 \times 3 + 2 \times 3^2 + \dots\dots\dots\dots + n \times 3^n = \frac{(2n-1)3^{n+1} + 3}{4}$$

where n is an integer, $n \geq 1$.

d)



An arrow fired from ground level at a velocity of 40 m/s, is to strike the ground 80 metres away as shown in the diagram above.

i) Show that the vertical and horizontal displacement equations are given respectively by: $y = 40t\sin\theta - 5t^2$ and $x = 40t\cos\theta$. (Assume $g = 10$ m/s²) 2

ii) Hence, find the two angles at which the arrow can be fired. 4

END OF EXAM

Candidate Name/Number: ______________________________________

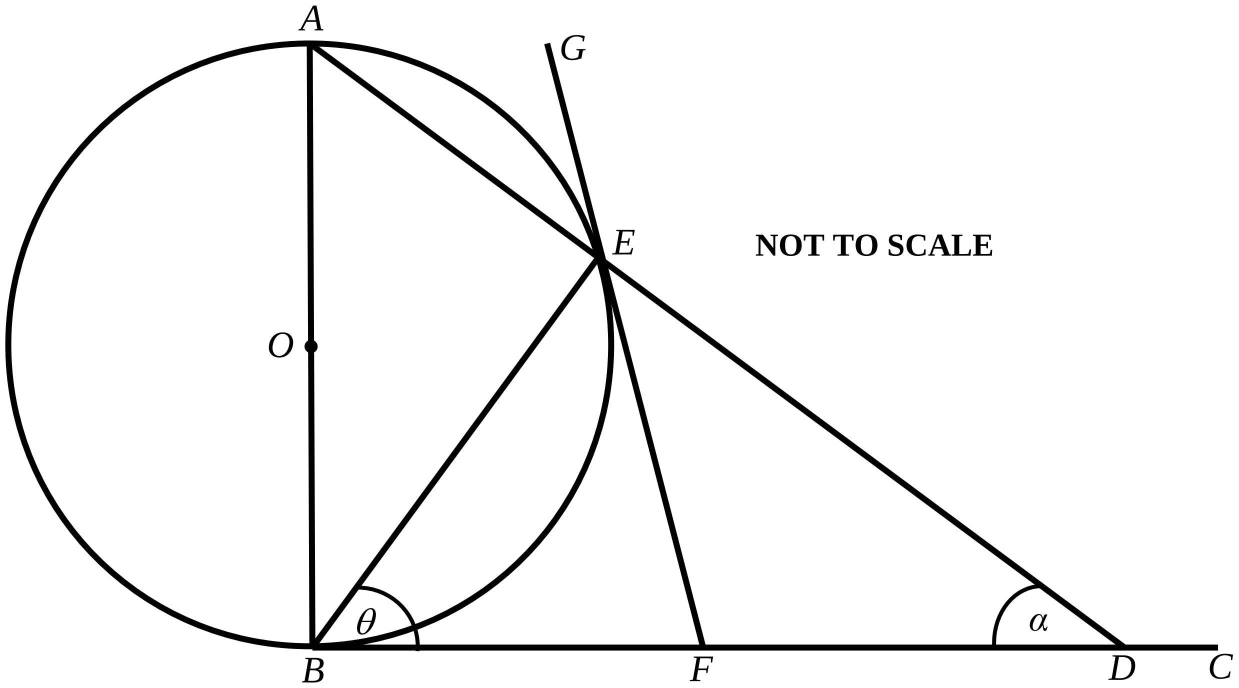
Multiple choice answer page. Fill in either A, B, C or D for questions 1-10.

This page must be handed in with your answer booklets

1.	
2.	
3.	
4.	
5.	

6.	
7.	
8.	
9.	
10.	

Question 14b (additional diagram)   Candidate Name/Number: ________________________________



Hand this page in inside the Question 14 booklet

STANDARD INTEGRALS

$$\int x^n\,dx = \frac{1}{n+1}x^{n+1}, \quad n \neq -1; \quad x \neq 0, \text{ if } n < 0$$

$$\int \frac{1}{x}\,dx = \ln x, \quad x > 0$$

$$\int e^{ax}\,dx = \frac{1}{a}e^{ax}, \quad a \neq 0$$

$$\int \cos ax\,dx = \frac{1}{a}\sin ax, \quad a \neq 0$$

$$\int \sin ax\,dx = -\frac{1}{a}\cos ax, \quad a \neq 0$$

$$\int \sec^2 ax\,dx = \frac{1}{a}\tan ax, \quad a \neq 0$$

$$\int \sec ax \tan ax\,dx = \frac{1}{a}\sec ax, \quad a \neq 0$$

$$\int \frac{1}{a^2 + x^2}\,dx = \frac{1}{a}\tan^{-1}\frac{x}{a}, \quad a \neq 0$$

$$\int \frac{1}{\sqrt{a^2 - x^2}}\,dx = \sin^{-1}\frac{x}{a}, \quad a > 0, \quad -a < x < a$$

$$\int \frac{1}{\sqrt{x^2 - a^2}}\,dx = \ln\left(x + \sqrt{x^2 - a^2}\right), \quad x > a > 0$$

$$\int \frac{1}{\sqrt{x^2 + a^2}}\,dx = \ln\left(x + \sqrt{x^2 + a^2}\right)$$

NOTE : $\ln x = \log_e x, \quad x > 0$

CHS YEAR 12 EXTENSION 1 2014	TRIAL HSC SOLUTIONS

Multiple Choice Section:

1.C	2.D	3.A	4.D	5.A
6.B	7.A	8.B	9.B	10.C

Question 1.

The remainder is given by $P(-1)$
$=2(-1)+1-5(-1)-3=1$ ---------- $\boxed{C}$

Question 2.

$$\int \frac{1}{3+x^2}\,dx = \int \frac{1}{\sqrt{3}^2+x^2}\,dx$$

$$\frac{1}{\sqrt{3}}\tan^{-1}\frac{x}{\sqrt{3}}+c \quad \text{----------} \boxed{D}$$

Question 3.

$$f(x):\ y=\frac{2}{x+1} \to f^{-1}(x): x=\frac{2}{y+1}$$

$$\therefore\ xy+x=2 \to xy=2-x$$

$$\therefore\ y=\frac{2-x}{x} \quad \text{------------} \boxed{A}$$

Question 4.

$$\lim_{x\to 0}\frac{\sin 3x}{x}=3\lim_{x\to 0}\frac{\sin 3x}{3x}$$

$$=3\times 1=3 \quad \text{------------} \boxed{D}$$

Question 5.

$$3^{x-1}=5 \to x-1=\log_3 5$$
$$\therefore x=1+\log_3 5$$
$$=1+\frac{\ln 5}{\ln 3} \quad \text{-----------} \boxed{A}$$

Question 6.

$$\sin\theta=\frac{\sqrt{3}}{2} \to \text{Acute angle } \theta=\frac{\pi}{3}$$

$$\therefore\ \theta=k\pi+(-1)^k\frac{\pi}{3} \quad \text{------------} \boxed{B}$$

Question 7.

$$\text{product of roots}=\frac{-d}{a}$$

$$\therefore -1\times 2\times\alpha=\frac{-(-8)}{2}=4$$

$$\therefore\ \alpha=-2 \quad \text{------------} \boxed{A}$$

Question 8.

$$f(x)=\sin^2(3-x) \to f'(x)=-2\sin(3-x)\cos(3-x)$$

$$\therefore\ f'(0)=-2\sin(3)\cos(3) \quad \text{------------} \boxed{B}$$

Question 9.

Domain: $-1\le \frac{x}{2}\le 1 \to -2\le x\le 2$

Range: $0\le y\le 3\pi$ ------------ $\boxed{B}$

Question 10.

$\angle PBA=90^\circ$ $\{\angle$ *in semi-circle*$\}$

Given $\angle APB=40^\circ \to \angle PAB=50^\circ$ $\{\angle$ *sum* $\Delta APB\}$

------------ $\boxed{C}$

Question 11.

a) CV1: $2x=3(x-1) \to x=3$

CV2: $x=1$

On testing $x<1$ and $x>3$, note $x\ne 1$

Question 11 continued.

b) $x - 2y + 3 = 0 \;\rightarrow\; m_1 = \dfrac{1}{2}$

$3x + y + 6 = 0 \;\rightarrow\; m_2 = -3$

$$\therefore\ \tan\theta = \left|\frac{1/2 - (-3)}{1 + 1/2 \times (-3)}\right| = 7$$

$$\therefore\ \theta = 81^\circ 52' \;(NOTE:\ accept\ 82^\circ)$$

c) $1 + \tan\theta\tan\dfrac{\theta}{2} = 1 + \dfrac{2t}{1-t^2} \times t$

$$= \frac{1 - t^2 + 2t^2}{1 - t^2}$$

$$= \frac{1 + t^2}{1 - t^2}$$

d) $\cos 2x = 1 - 2\sin^2 x \;\rightarrow\; \sin^2 x = \dfrac{1}{2}(1 - \cos 2x)$

$$\therefore\ I = \frac{1}{2}\int 1 - \cos 2x\,dx$$

$$= \frac{1}{2}\left(x - \frac{1}{2}\sin 2x\right) + c$$

$$= \frac{x}{2} - \frac{1}{4}\sin 2x + c$$

e) Let $P(x) = 3x + \ln x - 1 \;\rightarrow\; P'(x) = 3 + \dfrac{1}{x}$

$$a_1 = a_0 - \frac{P(a_0)}{P'(a_0)}\ (Newton's\ method)$$

$\boxed{P(0.4) = -0.7163;\ P'(0.4) = 5.5}$

$$\therefore\ a_1 = 0.4 - \frac{-0.7163}{5.5}$$

$$\approx 0.53\,(2\,dp)$$

f) $2\cos^2\theta - 1 = \cos\theta$

$\therefore\ 2\cos^2\theta - \cos\theta - 1 = 0$

$(2\cos\theta + 1)(\cos\theta - 1) = 0$

$\therefore\ \cos\theta = -\dfrac{1}{2}$ and $\cos\theta = 1$

$$\therefore\ \theta = \frac{2\pi}{3}, \frac{4\pi}{3}, 0, 2\pi$$

g) i) $\sin(A - B) = \sin A\cos B - \sin B\cos A$

ii) $\sin(45^\circ - 30^\circ) = \sin 45^\circ\cos 30^\circ - \sin 30^\circ\cos 45^\circ$

$$\therefore\ \sin 15^\circ = \frac{1}{\sqrt{2}} \times \frac{\sqrt{3}}{2} - \frac{1}{2} \times \frac{1}{\sqrt{2}}$$

$$= \frac{\sqrt{3} - 1}{2\sqrt{2}} \times \frac{\sqrt{2}}{\sqrt{2}}$$

$$= \frac{\sqrt{6} - \sqrt{2}}{4}.$$

$\left[Note:\ \text{Could also use } \sin\left(60^\circ - 45^\circ\right)\right]$

Question 12.

a) $x = 3t \;\rightarrow\; t = \dfrac{x}{3}$

$\therefore\ y = 4t^2 \;\rightarrow\; y = 4\left(\dfrac{x}{3}\right)^2$

$$\therefore\ y = \frac{4x^2}{9}\ \ or\ \ 4x^2 = 9y$$

b) For $\sin^{-1} x$: $\boxed{D: -1 \le x \le 1}$

and for $\ln(x)$: $\boxed{D:\ x > 0}$

$\therefore$ for $\ln\left(\sin^{-1} x\right)$: $\boxed{D: 0 < x \le 1}$

c) i) A function is odd if $f(-x) = -f(x)$

$f(x) = \dfrac{3x}{x^2 - 1} \;\rightarrow\; f(-x) = \dfrac{3(-x)}{(-x)^2 - 1}$

$$= \frac{-3x}{x^2 - 1} = -f(x)$$

ii) A function is decreasing if $f'(x) < 0$ for all x

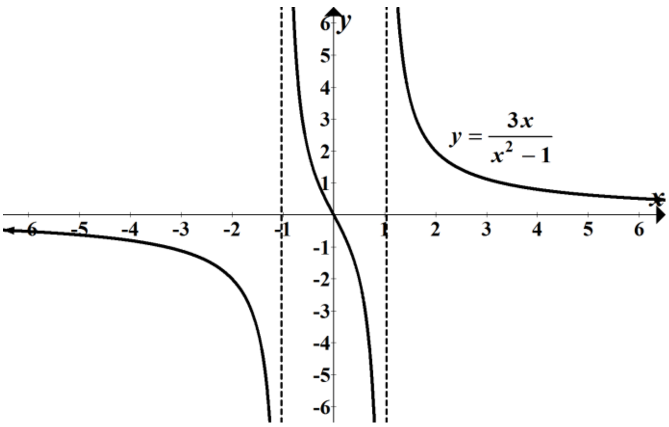
$$f'(x) = \frac{\left(x^2 - 1\right) \times 3 - 3x \times 2x}{\left(x^2 - 1\right)^2}$$

$$= \frac{-3\left(x^2 + 1\right)}{\left(x^2 - 1\right)^2} < 0 \ \text{ for all } x$$

Question 12 continued:

c) iii) Vertical asymptotes at $x = \pm 1$

Horizontal asymptote at $y = 0 \ (x-axis)$



d) $u = x^2 \ \rightarrow \ du = 2x\,dx$

When $x = \sqrt{3}, u = 1;\ x = \sqrt{11}, u = 9$

$$\therefore \ I = \frac{1}{2}\int_1^9 \frac{1}{\sqrt{u}}\,du = \frac{1}{2}\int_1^9 u^{-1/2}\,du$$

$$= \frac{1}{2}\left[2u^{1/2}\right]_1^9 = 3 - 1 = 2$$

e) $A = \pi r^2;\ r = \sqrt{t+1} - 1 \ \rightarrow \ r + 1 = \sqrt{t+1}$

$$\frac{dA}{dt} = \frac{dA}{dr} \times \frac{dr}{dt} \text{ [chain rule]}$$

$$= 2\pi r \times \frac{1}{2}(t+1)^{-1/2}$$

$$= \pi r \times \frac{1}{\sqrt{(t+1)}}$$

$$\therefore \ \frac{dA}{dt} = \pi r \times \frac{1}{r+1}$$

and when $r = 1$:

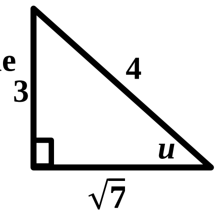
$$\therefore \ \frac{dA}{dt} = \pi \times 1 \times \frac{1}{1+1} = \frac{\pi}{2}\,km/h$$

Question 13.

a) Let $u = \sin^{-1}\left(\frac{3}{4}\right)$ and evaluate $\cos u$

$$u = \sin^{-1}\left(\frac{3}{4}\right) \ \rightarrow \ \sin u = \frac{3}{4}$$

$\cos u = \sqrt{1 - \sin^2 u}$ **or use a triangle**



$$= \sqrt{1 - 9/16} = \sqrt{7/16}$$

$$\therefore \ \cos u = \frac{\sqrt{7}}{4} = \cos\left(\sin^{-1}\left(\frac{3}{4}\right)\right)$$

b) i) Let $\cos x - \sin x = A\cos(x + \alpha)$

$$= A\cos\alpha\cos x - A\sin\alpha\sin x$$

Equating coefficients of $\cos x$ and $\sin x$ gives:

$A\cos\alpha = 1$ ---- $\boxed{1}$ and $A\sin\alpha = 1$ ---- $\boxed{2}$

$\boxed{1}^2 + \boxed{2}^2 \ \rightarrow \ A^2\left(\cos^2 x + \sin^2 x\right) = 2$

$\therefore \ A^2 = 2 \ \rightarrow \ A = \sqrt{2}$

Using $\boxed{1}$ with $A = \sqrt{2} \ \rightarrow \ \cos\alpha = \frac{1}{\sqrt{2}}$

$$\therefore \ \alpha = \frac{\pi}{4}$$

ii) $\cos x - \sin x = \frac{1}{\sqrt{2}}$

$$\therefore \ \sqrt{2}\cos\left(x + \frac{\pi}{4}\right) = \frac{1}{\sqrt{2}} \ \rightarrow \ \cos\left(x + \frac{\pi}{4}\right) = \frac{1}{2}$$

$$\therefore \ x + \frac{\pi}{4} = \frac{\pi}{3}, \frac{5\pi}{3}$$

hence $x = \frac{\pi}{12}, \frac{17\pi}{12}$

c) i) $T = 20 + Ae^{-kt} \ \rightarrow \ Ae^{-kt} = T - 20$

$$LHS = \frac{d\left(20 + Ae^{-kt}\right)}{dt}$$

$$= -kAe^{-kt}$$

$$= -k\left(T - 20\right) = RHS$$

cii) when $t=0, T=180°$; $t=5, T=120°$

$\therefore\ 180 = 20 + Ae^0 \rightarrow A = 160$

also $120 = 20 + 160e^{-5k}$

$\therefore\ \frac{10}{16} = e^{-5k} \rightarrow -5k = \ln\left(\frac{5}{8}\right)$

$\therefore\ k = -\frac{1}{5}\ln\left(\frac{5}{8}\right) \rightarrow k \approx 0.094$

ii) $60 = 20 + 160e^{-kt}$

$\therefore\ \frac{1}{4} = e^{-0.094t}$

$\therefore\ \ln\left(\frac{1}{4}\right) = -0.094t \rightarrow t \approx 15$ minutes

d) i) $y = \frac{x^2}{4a} \rightarrow m = \frac{dy}{dx} = \frac{x}{2a}$

At P: $m = \frac{2ap}{2a} = p$

hence $y - ap^2 = p(x - 2ap)$

$y - ap^2 = px - 2ap^2$

$\therefore\ y = px - ap^2$

ii) $y = px - ap^2$ ------ $\boxed{1}$

at Q: $y = qx - aq^2$ ------ $\boxed{2}$

$\boxed{1} = \boxed{2} \rightarrow px - ap^2 = qx - aq^2$

$\therefore\ px - qx = ap^2 - aq^2$

$x(p-q) = a(p-q)(p+q)$

$\therefore\ x = a(p+q)$

hence $y = p \times a(p+q) - ap^2 = apq$

$\therefore\ T\left[a(p+q),\ apq\right]$

iii) since $\angle POQ = 90°$, $m_{OP} \times m_{OQ} = -1$

$\therefore\ \frac{ap^2}{2ap} \times \frac{aq^2}{2aq} = -1$

$\therefore\ \frac{p}{2} \times \frac{q}{2} = -1 \rightarrow pq = -4$

iv) Since $pq = -4$

$y = apq \rightarrow y = -4a$

which is the locus of T.

Question 14.

a) $P(x) = 4x^3 + kx + 6$

$P(-3) = 0 \rightarrow -108 - 3k + 6 = 0$

$\therefore\ k = -34$

so $P(x) = 4x^3 - 34x + 6$

$\therefore\ 4x^3 - 34x + 6 = (x+3)(4x^2 + bx + 2)$

on equating coefficients of x^2:

$0 = b + 12 \rightarrow b = -12$

$\therefore\ P(x) = (x+3)(4x^2 - 12x + 2)$

Note:[or use long division]

b) i) $\angle AEB = 90°$ [$\angle$ in semi-circle]

$BF = FE$ [tangents from external point equal]

$\therefore\ \Delta FBE$ is isosceles

$\therefore\ \angle FEB = \theta$ [$\angle$'s opposite equal sides in isos Δ]

and since $\angle BED = 90°$, then

$\angle FED = 90° - \theta = \frac{\pi}{2} - \theta$.

ii) Using ΔABD: $\alpha = 180° - \angle ABD - \angle BAD$

$= 180° - 90° - \theta$

$= 90° - \theta$ or $\left(\frac{\pi}{2} - \theta\right)$

$\therefore\ \Delta FED$ is isosceles with $FE = FD$

[sides opposite equal $\angle$'s in isos Δ]

and also since $FE = BF$, then $BF = FD$.

c) $1 \times 3 + 2 \times 3^2 + \ldots\ldots\ldots\ldots + n \times 3^n = \frac{(2n-1)3^{n+1} + 3}{4}$

When $n = 1$: $LHS = 1 \times 3 = 3$

$RHS = \frac{(2 \times 1 - 1) \times 3^2 + 3}{4}$

$= \frac{9 + 3}{4} = 3$ $\therefore$ true for n=1.

Question 14 continued:

Assume true for $n = k$: i.e.

$$1 \times 3 + 2 \times 3^2 + \ldots\ldots\ldots\ldots + k \times 3^k = \frac{(2k-1)3^{k+1}+3}{4}$$

Prove true for $n = k$:

i.e. $\boxed{S_k + T_{k+1} = S_{k+1}}$

$$S_k = \frac{(2k-1)3^{k+1}+3}{4};\ S_{k+1} = \frac{(2k+1)3^{k+2}+3}{4};\ T_{k+1} = (k+1)\times 3^{k+1}$$

$$LHS = \frac{(2k-1)3^{k+1}+3}{4} + (k+1)\times 3^{k+1}$$

$$= \frac{(2k-1)3^{k+1}+3 + 4(k+1)\times 3^{k+1}}{4}$$

$$= \frac{3^{k+1}(2k-1+4(k+1))+3}{4}$$

$$= \frac{3^{k+1}(6k+3)+3}{4}$$

$$= \frac{(2k+1)\times 3\times 3^{k+1}+3}{4}$$

$$= \frac{(2k+1)\times 3^{k+2}+3}{4}$$

$$= S_{k+1} = RHS$$

$\therefore$ If true for $n = k$, then true for $n = k + 1$.

Hence by the principle of mathematical induction, the result is true for all $n \geq 1$.

d) i) $t = 0,\ y = 0,\ \dot{y} = 40\sin\theta,\ x = 0,\ \dot{x} = 40\cos\theta$

Vertically: $\ddot{y} = -10 \rightarrow \dot{y} = -10t + c$

$t = 0,\ \dot{y} = 40\sin\theta \rightarrow c = 40\sin\theta$

$\therefore \dot{y} = 40\sin\theta - 10t$

$\therefore y = 40t\sin\theta - 5t^2 + c$

$0 = 40\sin\theta\times 0 - 5\times 0 + c \rightarrow c = 0$

$\therefore y = 40t\sin\theta - 5t^2$

Horizontally: $\ddot{x} = 0 \rightarrow \dot{x} = 40\cos\theta$

$\therefore x = 40t\cos\theta + c$

$0 = 40\cos\theta\times 0 + c \rightarrow c = 0$

$\therefore x = 40t\cos\theta$

ii) When $y = 0,\ 0 = t(40\sin\theta - 5t)$

$\therefore$ when it strikes the ground $t = 8\sin\theta$

$\therefore$ when $t = 8\sin\theta,\ x = 80$

$\therefore 80 = 40\cos\theta\times 8\sin\theta$

$$\frac{1}{2} = 2\sin\theta\cos\theta$$

$$\frac{1}{2} = \sin 2\theta$$

$\therefore 2\theta = 30°,\ 180° - 30°$

hence the two angles are $15°$ and $75°$.