2012

Higher School Certificate Trial Examination

Mathematics Extension 2

General Instructions

- Reading time – 5 minutes
- Working time – 3 hours
- Board approved calculators may be used.
- Write using black or blue pen
- A table of standard integrals is provided
- All necessary working should be shown in every question

Total marks – 100

Section 1 - 10 marks

- Attempt Questions 1 – 10
- Circle the correct option on the sheet

Section 2 - 90 marks

- Attempt Questions 11 - 16
- All questions are of equal value
- Answer each question in a separate answer booklet

MCQ	Q1	Q2	Q3	Q4	Q5	Q6	TOTAL

NAME:.. TEACHER:...

SECTION 1 - [10 Marks]

Allow about 15 minutes for this section

Circle the correct option that best answers the question.

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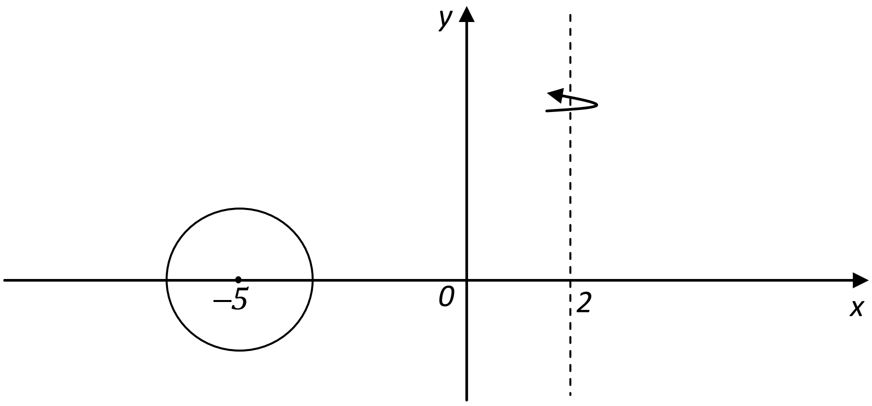
1. On an Argand diagram, the points *A* and *B* represent the complex numbers $z_1 = -2i$ and $z_2 = 1 - \sqrt{3}i$. Which of the following statements is true?

A. $arg(z_1 + z_2) = -\frac{5\pi}{12}$ B. $|z_1 - z_2| = 2 + \sqrt{3}$

C. $arg\left(\frac{z_1}{z_2}\right) = \frac{\pi}{6}$ D. $arg(z_1 z_2) = -\frac{\pi}{3}$

2. The region bounded by the circle $(x+5)^2 + y^2 = 4$ is rotated about the line $x = 2$.



The volume of the solid of revolution is

A. $56\pi^2$ *cubic units* B. $28\pi^2$ *cubic units*

C. $50\pi^2$ *cubic units* D. $7\pi^2$ *cubic units*

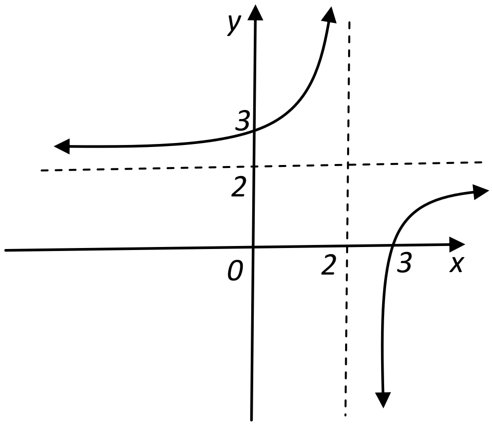
3. If the line $y = mx + k$ is a tangent to the hyperbola $xy = c^2$, which of the following statements is true?

A. $k^2 = -4mc^2$ B. $k^2 = 4mc^2$ C. $k = 4mc$ D. $c^2 = 4mk$

4. The value of $\int_{0}^{\frac{\pi}{2}} \frac{\cos x}{\sin x + \cos x}\, dx$ is equal to

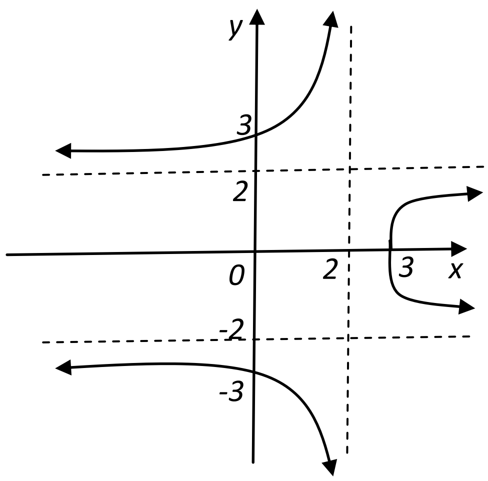
A. 0 B. π C. $\frac{\pi}{2}$ D. $\frac{\pi}{4}$

5. The graph of $y = f(x)$ is given below.

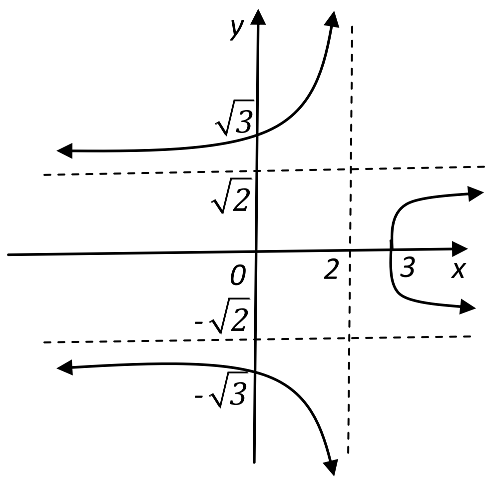


The graph of $y^2 = f(x)$ is

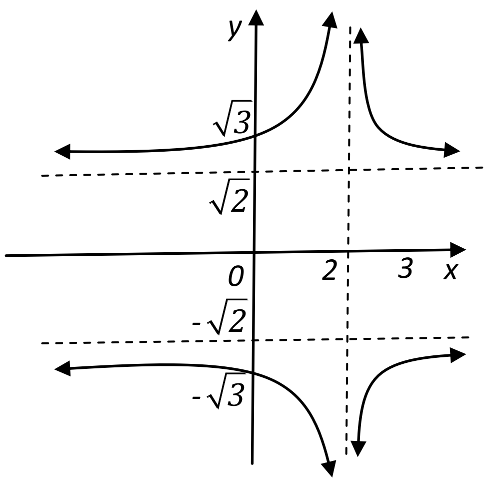
A.



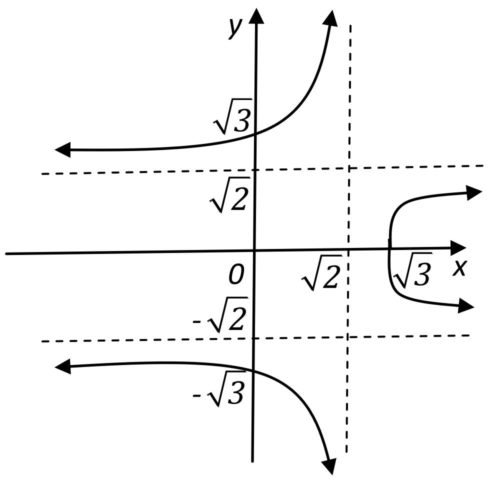
B.



C.



D.



6. The polynomial equation $P(z)=0$ has one complex coefficient. Three of its roots are $z=1-i$, $z=2-3i$ and $z=0$. The minimum degree of $P(z)=0$ is

A. 1 B. 2 C. 3 D. 4

7. The algebraic fraction $\dfrac{x+1}{5(x+h)^2}$, where *h* is a non-zero real number can be written in partial fraction from, where *A* and *B* are real numbers, as

A. $\dfrac{A}{x+h}+\dfrac{B}{x+h}$

B. $\dfrac{A}{5x+h}+\dfrac{B}{(x+h)^2}$

C. $\dfrac{A}{x+h}+\dfrac{B}{(x+h)^2}$

D. $\dfrac{A}{5(x+h)}+\dfrac{B}{x+h}$

8. The value of $\displaystyle\int_{-1}^{1}\frac{1}{1+e^{-x}}\,dx$ is

A. $\dfrac{1}{2}$ B. 1 C. $ln(1+e)$ D. $2\,ln(1+e)$

9. A particle of unit mass falls from rest from the top of a cliff in a medium where the resistive force is kv^2. The distance fallen through when it reaches a speed half its terminal velocity is given by

A. $x=\dfrac{1}{2k}\,ln\left[\dfrac{3}{4}\right]$

B. $x=\dfrac{1}{2k}\,ln\left[\dfrac{4}{3}\right]$

C. $x=\dfrac{1}{2k}\,ln\left[\dfrac{5}{4}\right]$

D. $x=\dfrac{1}{2k}\,ln\left[\dfrac{4}{5}\right]$

10. *P* is a variable point on the hyperbola $4x^2-y^2=4$. If *m* is the gradient of the tangent to the hyperbola at *P*, then *m* is any real number such that

A. $-2<m<2$

B. $-2\le m\le 2$

C. $m<-2$ or $m>2$

D. $m\le -2$ or $m\ge 2$

SECTION 2 - [90 marks]

Use a separate answer booklet for each question

Allow about 2 hours and 45 minutes for this section

Question 11	**Start on a new answer booklet**	**Marks**

a) Find $\int \frac{\sqrt{1-x^2}}{\sqrt{1-x}}\, dx$ **2**

b) Evaluate in the simplest form $\int_0^4 \frac{8-2x}{(1+x)(4+x^2)}\, dx$ **3**

c) i) Use the substitution $u = \frac{\pi}{4} - x$, to show

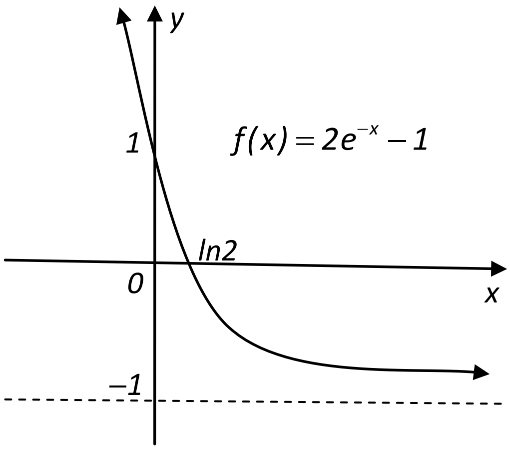
$$\int_0^{\frac{\pi}{4}} ln\left(1+\tan x\right) dx = \int_0^{\frac{\pi}{4}} ln\left(\frac{2}{1+\tan x}\right) dx$$ **3**

ii) Hence find the exact value of $\int_0^{\frac{\pi}{4}} ln\left(1+\tan x\right) dx$ **2**

d) Consider the integral $I_n = \int_0^1 \sqrt{x}\left(1-x\right)^n dx, n = 0,1,2,3,....$

i) Show that $I_n = \left(\frac{2n}{2n+3}\right) I_{n-1}$, $n = 1,2,3,.......$ **3**

ii) Hence evaluate $I_3 = \int_0^1 \sqrt{x}\left(1-x\right)^3 dx$ **2** a)



The diagram shows the graph of $f(x)=2e^{-x}-1$.

On separate diagrams, sketch the following graphs, showing the intercepts on the axes and the equations of any asymptotes:

i) $y=|f(x)|$ ii) $y=[f(x)]^2$

iii) $y=\dfrac{1}{f(x)}$ iv) $y=\ln[f(x)]$ **5**

b) The ellipse $\dfrac{x^2}{a^2}+\dfrac{y^2}{b^2}=1$, where $a>b>0$, has eccentricity $e=\dfrac{1}{2}$.

The point $P(2,3)$ lies on the ellipse.

i) Find the values of *a* and *b*. **3**

ii) Sketch the graph of the ellipse, showing clearly the intercepts on the axes, the coordinates of the foci and the equations of the directrices. **3**

c) Consider the curve defined by the equation $3x^2+y^2-2xy-8x+2=0$.

i) Show that $\dfrac{dy}{dx}=\dfrac{3x-y-4}{x-y}$. **2**

ii) Find the coordinates of the points on the curve where the tangent to the curve is parallel to the line $y=2x$. **2**

Question 13 **Start on a new answer booklet** **Marks**

a) The fixed complex number α is such that $0 < arg\,\alpha < \frac{\pi}{2}$. In an Argand diagram α is represented by the point *A* while $i\alpha$ is represented by the point *B*.

z is a variable complex number which is represented by the point *P*.

i) Draw a diagram showing *A*, *B* and the locus of *P* if $\left| z-\alpha \right| = \left| z - i\alpha \right|$ **1**

ii) Draw a diagram showing *A*, *B* and the locus of *P* if $arg\left(z-\alpha\right) = arg\left(i\alpha\right)$ **1**

iii) Find in terms of α the complex number represented by the point of intersection of the two loci in (i) and (ii). **1**

b) It is given that $z = cos\theta + i\,sin\theta$, where $0 < arg\,z < \frac{\pi}{2}$.

i) Show that $z+1 = 2cos\frac{\theta}{2}\left(cos\frac{\theta}{2} + i\,sin\frac{\theta}{2}\right)$ and express $z-1$ in the modulus-argument form. **3**

ii) Hence show that $Re\left(\frac{z-1}{z+1}\right) = 0$. **1**

c) i) Express the roots of the equation $z^2 + 4z + 8 = 0$ in the form $a+ib$, where *a* and *b* are real. **1**

ii) Hence express the roots of the equation $z^2 + 4z + 8 = 0$ in the modulus-argument form. **2**

d) *P* and *Q* are points on the curve $y = x^4 + 4x^3$ with x-coordinates α *and* β respectively. The line $y = mx + b$ is a tangent to the curve at both points *P* and *Q.*

i) Explain why the equation $x^4 + 4x^3 - mx - b = 0$ has roots α, α, β *and* β . **1**

ii) Use the relationships between the roots and the coefficients of this equation to find the values of *m* and *b*. **4**

a) $P\left(cp,\dfrac{c}{p}\right)$, $Q\left(cq,\dfrac{c}{q}\right)$ are points on the rectangular hyperbola $xy = c^2$.

Tangents to the rectangular hyperbola at *P* and *Q* intersect at the point $R\left(X,Y\right)$.

i) Show that the tangent to the rectangular hyperbola at $\left(ct,\dfrac{c}{t}\right)$ has equation $x + t^2y = 2ct$. **2**

ii) Show that $X = \dfrac{2cpq}{p+q}$, $Y = \dfrac{2c}{p+q}$. **2**

iii) If *P* and *Q* are variable points on the rectangular hyperbola such that $p^2 + q^2 = 2$, find the equation of the locus of *R*. **3**

b) A particle *P* of mass *m kg* is projected vertically upwards with speed *U m/s* in a medium in which the resistance to motion has magnitude $\dfrac{1}{10}mv^2$ when the speed of the particle is *v m/s*. After *t seconds* the particle has height *x metres*, velocity *v m/s* and acceleration $a\ m/s^2$.

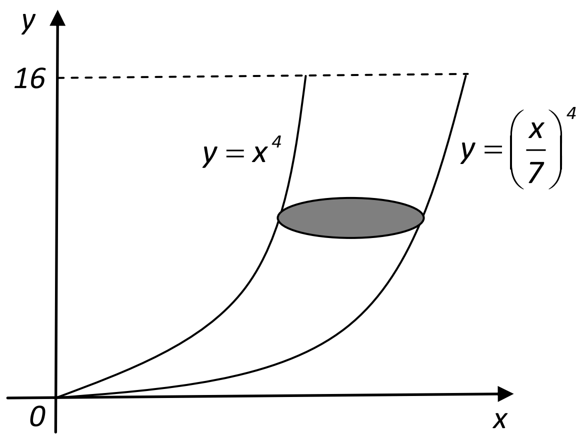
i) Draw a diagram showing forces acting on the particle *P*, and hence show that

$$a = -\left(\frac{v^2 + 100}{10}\right) .$$ **2**

ii) Find, in terms of *U*, the time taken for the particle to reach the maximum height. **3**

iii) Find the maximum height in terms of *U*. **3**

a) A mould for a drinking horn is bounded by the curves $y = x^4$ and $y = \left(\frac{x}{7}\right)^4$ between $y = 0$ and $y = 16$.



Every cross-section perpendicular to the y-axis is a circle. All measurements are in cm.

Find the capacity of the drinking horn in litres, correct to three significant figures. **5**

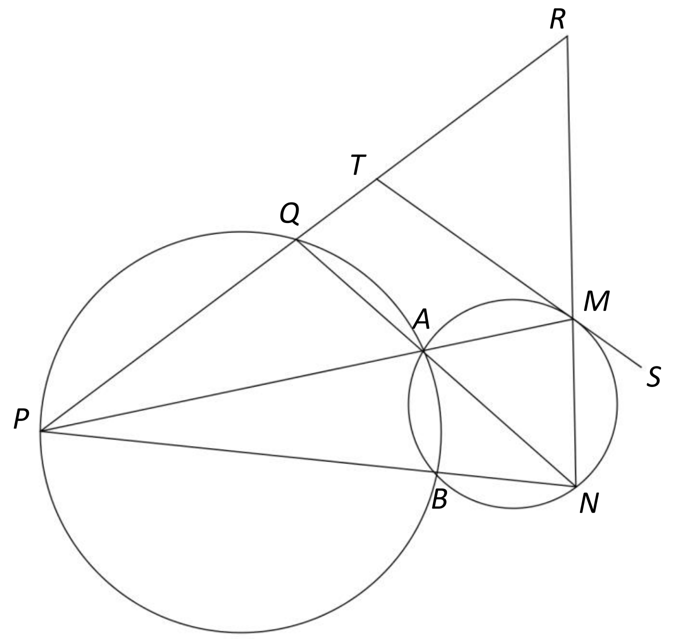
b) A sequence of numbers T_n $, n = 1, 2, 3, ...$ is defined by $T_1 = 2, T_2 = 0$ and $T_n = 2T_{n-1} - 2T_{n-2}$ for $n = 3, 4, 5, ...$.

Use mathematical induction to prove that $T_n = \left(\sqrt{2}\right)^{n+2} \cos\left(\frac{n\pi}{4}\right)$ for $n = 1, 2, 3, ...$ **5**

c) The hyperbola $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1, \left(0 < a < b\right)$ has eccentricity *e*. *S* is the focus of the hyperbola on the positive *x*- axis. The line through *S* perpendicular to the *x*-axis intersects the hyperbola at *P* and *Q*.

i) Show that $PQ = \frac{2b^2}{a}$ **2**

ii) If P and Q have coordinates $\left(9, 24\right)$ and $\left(9, -24\right)$ respectively, write down two equations in a and b, then solve these equations algebraically to show that $a = 3$ and $b = 6\sqrt{2}$. **3** a)



In the diagram, the two circles intersect at *A* and *B*. *P* is a point on one circle. *PA* and *PB* produced meet the other circle at *M* and *N* respectively. *NA* produced meets the first circle at *Q*. *PQ* and *NM* produced meet at *R*. The tangent at *M* to the second circle meets *PR* at *T*.

i) Copy the diagram. Show that *QAMR* is a cyclic quadrilateral. **2**

ii) Show that *TM = TR*. **4**

b) $P(x)=x^4-2x^3+3x^2-4x+1$ and the equation $P(x)=0$ has roots α,β,γ and δ .

i) Show that the equation $P(x)=0$ has no integer roots. **1**

ii) Shaw that $P(x)=0$ has a real root between 0 and 1. **1**

iii) Show that $\alpha^2+\beta^2+\gamma^2+\delta^2=-2$. **2**

iv) Hence find the number of real roots of the equation $P(x)=0$,

giving reasons. **2**

c) The polynomial $P(x)=x^3-6x^2+9x+c$ has a double zero. Find any possible

values of the real number *c*. **3**

End of paper.

STANDARD INTEGRALS

$$\int x^n\,dx = \frac{1}{n+1}x^{n+1}, \quad n \neq -1; \quad x \neq 0, \text{ if } n < 0$$

$$\int \frac{1}{x}\,dx = \ln x, \quad x > 0$$

$$\int e^{ax}\,dx = \frac{1}{a}e^{ax}, \quad a \neq 0$$

$$\int \cos ax\,dx = \frac{1}{a}\sin ax, \quad a \neq 0$$

$$\int \sin ax\,dx = -\frac{1}{a}\cos ax, \quad a \neq 0$$

$$\int \sec^2 ax\,dx = \frac{1}{a}\tan ax, \quad a \neq 0$$

$$\int \sec ax \tan ax\,dx = \frac{1}{a}\sec ax, \quad a \neq 0$$

$$\int \frac{1}{a^2 + x^2}\,dx = \frac{1}{a}\tan^{-1}\frac{x}{a}, \quad a \neq 0$$

$$\int \frac{1}{\sqrt{a^2 - x^2}}\,dx = \sin^{-1}\frac{x}{a}, \quad a > 0, \quad -a < x < a$$

$$\int \frac{1}{\sqrt{x^2 - a^2}}\,dx = \ln\left(x + \sqrt{x^2 - a^2}\right), \quad x > a > 0$$

$$\int \frac{1}{\sqrt{x^2 + a^2}}\,dx = \ln\left(x + \sqrt{x^2 + a^2}\right)$$

TRIAL HSC 2012 - SOLUTIONS

SECTION 1 - MCQ

1.	**A**	**2.**	**A**	**3.**	**A**	**4.**	**D**	**5.**	**B**
6.	**C**	**7.**	**C**	**8.**	**B**	**9.**	**B**	**10.**	**C**

SECTION 2

QUESTION 11

a)

$$\int \frac{\sqrt{1-x^2}}{\sqrt{1-x}}\,dx = \int \sqrt{1+x}\quad dx$$
$$= \tfrac{2}{3}(1+x)^{\frac{3}{2}} + c$$

b)

$$\frac{8-2x}{(1+x)(4+x^2)} \equiv \frac{a}{1+x} + \frac{bx+c}{4+x^2}$$
$$8-2x \equiv a(4+x^2) + (bx+c)(1+x)$$

sub. $x=-1$: $\quad 10 = 5a \Rightarrow a = 2$

equate coeffs of x^2: $\quad 0 = a + b \Rightarrow b = -2$

sub. $x = 0$: $\quad 8 = 4a + c \Rightarrow c = 0$

$$\int_0^4 \frac{8-2x}{(1+x)(4+x^2)}\,dx = \int_0^4 \frac{2}{1+x} + \frac{-2x}{4+x^2}\,dx$$
$$= \left[2\ln(1+x) - \ln(4+x^2)\right]_0^4$$
$$= 2(\ln 5 - \ln 1) - (\ln 20 - \ln 4)$$
$$= \ln 5$$

c)

i. $u = \frac{\pi}{4} - x$

$du = -dx$

$x = 0 \Rightarrow u = \frac{\pi}{4}$

$x = \frac{\pi}{4} \Rightarrow u = 0$

$$\int_0^{\frac{\pi}{4}} \ln(1+\tan x)\quad dx = \int_{\frac{\pi}{4}}^0 \ln\left\{1+\tan\left(\tfrac{\pi}{4}-u\right)\right\}.\ -du$$
$$= \int_0^{\frac{\pi}{4}} \ln\left\{1 + \frac{1-\tan u}{1+\tan u}\right\} du$$
$$= \int_0^{\frac{\pi}{4}} \ln\left(\frac{2}{1+\tan u}\right) du$$
$$= \int_0^{\frac{\pi}{4}} \ln\left(\frac{2}{1+\tan x}\right) dx$$

ii.

$$\int_0^{\frac{\pi}{4}} \ln(1+\tan x)\quad dx = \int_0^{\frac{\pi}{4}} \left\{\ln 2 - \ln(1+\tan x)\right\}\quad dx$$
$$2\int_0^{\frac{\pi}{4}} \ln(1+\tan x)\quad dx = \int_0^{\frac{\pi}{4}} \ln 2\quad dx$$
$$= \tfrac{\pi}{4}\ln 2$$

$$\therefore \int_0^{\frac{\pi}{4}} \ln(1+\tan x)\quad dx = \tfrac{\pi}{8}\ln 2$$

d)

(i)

$$I_n = \int_0^1 \sqrt{x}\,(1-x)^n \; dx$$

$$= \left[\tfrac{2}{3}x^{\frac{3}{2}}(1-x)^n\right]_0^1 - \int_0^1 \tfrac{2}{3}x^{\frac{3}{2}}\left\{-n(1-x)^{n-1}\right\}\, dx$$

$$= 0 - \tfrac{2n}{3}\int_0^1 x^{\frac{1}{2}}(1-x-1)\,(1-x)^{n-1}\; dx$$

$$= -\tfrac{2n}{3}\int_0^1 \left\{x^{\frac{1}{2}}(1-x)^n - x^{\frac{1}{2}}(1-x)^{n-1}\right\}\, dx$$

$$= -\tfrac{2n}{3}\left(I_n - I_{n-1}\right)$$

$$\therefore \quad 3I_n = -2n\left(I_n - I_{n-1}\right)$$

$$3\,I_n = 2n\,I_{n-1} - 2n\,I_n$$

$$(2n+3)I_n = 2n\,I_{n-1}$$

$$I_n = \frac{2n}{(2n+3)}\; I_{n-1}$$

(ii)

$$I_3 = \tfrac{6}{9}\; I_2 \;=\; \tfrac{6}{9}\cdot\tfrac{4}{7}I_1 \;=\; \tfrac{6}{9}\cdot\tfrac{4}{7}\cdot\tfrac{2}{5}I_0$$

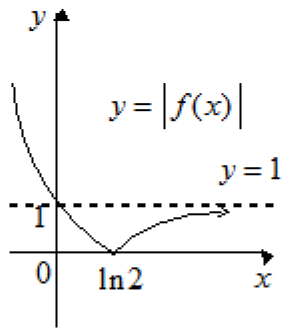
$$But \quad I_0 = \int_0^1 \sqrt{x}\; dx = \tfrac{2}{3}\left[x^{\frac{3}{2}}\right]_0^1 = \tfrac{2}{3}$$

$$\therefore I_3 = \tfrac{6}{9}\cdot\tfrac{4}{7}\cdot\tfrac{2}{5}\cdot\tfrac{2}{3} = \tfrac{32}{315}$$

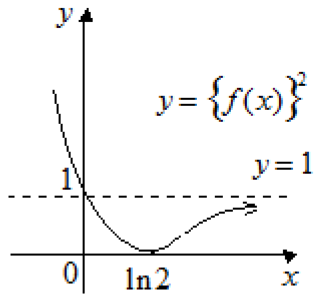
QUESTION 12

a)

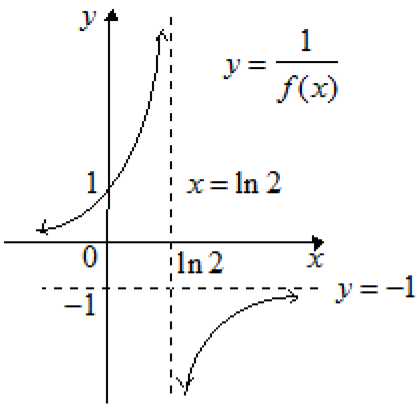
i.



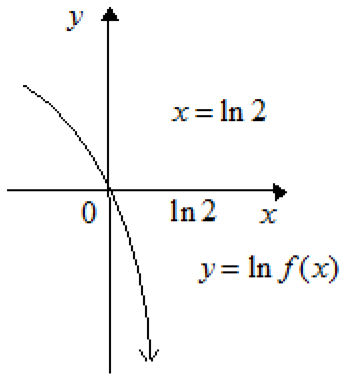
ii.



iii.



iv.



b)

(i) $e = \tfrac{1}{2} \Rightarrow b^2 = a^2\left(1-\tfrac{1}{4}\right) = \tfrac{3}{4}a^2$

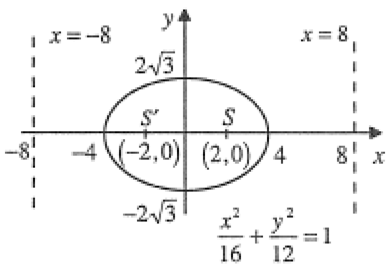
$P(2,3)$ on ellipse $\Rightarrow \dfrac{4}{a^2} + \dfrac{9}{b^2} = 1$

$\therefore \dfrac{4}{a^2} + \dfrac{12}{a^2} = 1$

$\therefore a^2 = 16,\; b^2 = 12$

$\therefore a = 4, \quad b = 2\sqrt{3}$

(ii)



c)

(i)

$$3x^2+y^2-2xy-8x+2=0$$

$$6x+2y\frac{dy}{dx}-2\left(1.\,y+x.\frac{dy}{dx}\right)-8+0=0$$

$$2(3x-y-4)-2(x-y)\frac{dy}{dx}=0$$

$$\therefore\ \frac{dy}{dx}=\frac{3x-y-4}{x-y}$$

(ii)
If the tangent to the curve at the point P is parallel to $y=2x$, then at P

$$\frac{dy}{dx}=2\ \Rightarrow\ \frac{3x-y-4}{x-y}=2$$

$$3x-y-4=2x-2y$$

$$y=4-x$$

$$\therefore\ 3x^2+(4-x)^2-2x(4-x)-8x+2=0$$

$$6x^2-24x+18=0$$

$$6(x-3)(x-1)=0$$

$\therefore$ at P, $y=4-x$, and $x=3$ or $x=1$.

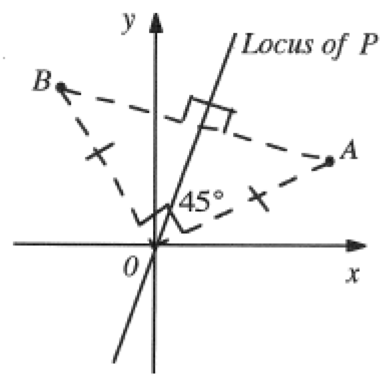
Hence the tangents at $(3,1)$ and $(1,3)$ are parallel to $y=2x$.

QUESTION 13

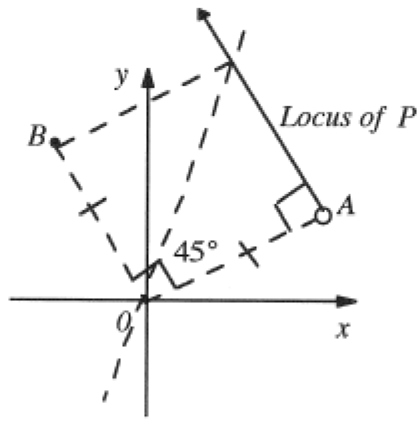
a)

Let $z=x+iy$, x, y real

(i) Locus of P is perpendicular bisector of AB.



(ii) Locus is ray from A parallel to $\overrightarrow{OB}$



(iii) If P is the point of intersection of these loci, $OAPB$ is a square and the diagonal $\overrightarrow{OP}$ represents the sum of α and $i\alpha$. Hence P represents $(1+i)\alpha$.

b)

(i)

$$z+1=1+\cos\theta+i\sin\theta$$
$$=2\cos^2\tfrac{\theta}{2}+i\left(2\sin\tfrac{\theta}{2}\cos\tfrac{\theta}{2}\right)$$
$$=2\cos\tfrac{\theta}{2}\left(\cos\tfrac{\theta}{2}+i\sin\tfrac{\theta}{2}\right)$$

$$z-1=-(1-\cos\theta)+i\sin\theta$$
$$=-2\sin^2\tfrac{\theta}{2}+i\left(2\sin\tfrac{\theta}{2}\cos\tfrac{\theta}{2}\right)$$
$$=2\sin\tfrac{\theta}{2}\left(-\sin\tfrac{\theta}{2}+i\cos\tfrac{\theta}{2}\right)$$
$$=2\sin\tfrac{\theta}{2}\left\{\cos\left(\tfrac{\pi}{2}+\tfrac{\theta}{2}\right)+i\sin\left(\tfrac{\pi}{2}+\tfrac{\theta}{2}\right)\right\}$$

(ii)

Then $\left|\dfrac{z-1}{z+1}\right|=\tan\tfrac{\theta}{2}$ and $\arg\left(\dfrac{z-1}{z+1}\right)=\left(\tfrac{\pi}{2}+\tfrac{\theta}{2}\right)-\tfrac{\theta}{2}=\tfrac{\pi}{2}$ $\Rightarrow$ $\dfrac{z-1}{z+1}=i\tan\tfrac{\theta}{2}$ $\therefore$ $\mathrm{Re}\left(\dfrac{z-1}{z+1}\right)=0$

c)

(i)

$$z^2 + 4z + 8 = 0$$
$$z^2 + 4z + 4 = -4$$
$$(z+2)^2 = -4$$
$$(z+2) = \pm 2i$$
$$z = -2 \pm 2i$$

(ii)

$$|z| = \sqrt{8} = 2\sqrt{2}$$
$$z = 2\sqrt{2}\left(-\tfrac{1}{\sqrt{2}} \pm \tfrac{1}{\sqrt{2}} i\right)$$

Hence roots are

$$2\sqrt{2}\left(\cos\tfrac{3\pi}{4} + i\sin\tfrac{3\pi}{4}\right),\quad 2\sqrt{2}\left\{\cos\left(-\tfrac{3\pi}{4}\right) + i\sin\left(-\tfrac{3\pi}{4}\right)\right\}$$

d)

i. Since the line $y = mx + b$ is tangent to the curve $y = x^4 + 4x^3$ at P where $x = \alpha$, and at Q where $x = \beta$, solving these equations simultaneously gives the equation $x^4 + 4x^3 - mx - b = 0$ with repeated roots $\alpha, \alpha, \beta, \beta$.

ii. Using the sum of roots is -4 and sum of products taken two at a time is 0 :

$$2\alpha + 2\beta = -4$$
$$\alpha^2 + \beta^2 + 4\alpha\beta = 0 \Rightarrow (\alpha+\beta)^2 + 2\alpha\beta = 0$$
$$\therefore \alpha + \beta = -2 \quad \text{and} \quad \alpha\beta = -2$$

Using the sum of products of roots taken three at a time is m, and the product of roots is $-b$:

$$m = 2\beta\alpha^2 + 2\alpha\beta^2 = 2\alpha\beta(\alpha+\beta) = 8$$
$$b = -\alpha^2\beta^2 = -4$$

QUESTION 14

a)

(i)

$$x = ct \Rightarrow \frac{dx}{dt} = c$$
$$y = \frac{c}{t} \Rightarrow \frac{dy}{dt} = -\frac{c}{t^2}$$
$$\therefore \frac{dy}{dx} = \frac{dy}{dt}\Big/\frac{dx}{dt} = -\frac{1}{t^2}$$

Hence tangent at $\left(ct, \frac{c}{t}\right)$ has gradient $-\frac{1}{t^2}$ and equation $x + t^2 y = k$ for some constant k.

$\left(ct, \frac{c}{t}\right)$ lies on the tangent $\Rightarrow ct + t^2\frac{c}{t} = k$

$\therefore k = 2ct$ and tangent has equation $x + t^2 y = 2ct$.

(ii) Where tangents at P, Q intersect

$$x + p^2 y = 2cp$$
$$x + q^2 y = 2cq$$
$$(p^2 - q^2)y = 2c(p-q)$$
$$(p-q)(p+q)\,y = 2c(p-q)$$

Also

$$(p^2 - q^2)x = 2c\,pq(p-q)$$
$$(p-q)(p+q)\,x = 2c\,pq\,(p-q)$$
$$\therefore p \neq q \Rightarrow X = \frac{2cpq}{p+q}, \quad Y = \frac{2c}{p+q}$$

(iii)

$$p^2 + q^2 = (p+q)^2 - 2pq$$
$$\therefore p^2 + q^2 = 2 \Rightarrow (p+q)^2 = 2(1+pq)$$

Hence at $R(X, Y)$

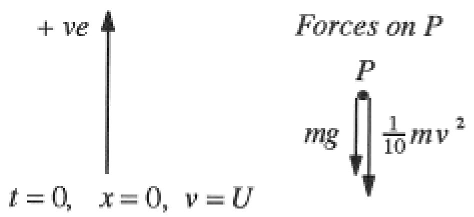
$$\frac{X}{Y} = pq \quad \text{and} \quad \frac{2c}{Y} = p + q$$

Hence the locus of R has equation

$$\frac{4c^2}{y^2} = 2\left(1 + \frac{x}{y}\right)$$
$$y^2 + xy = 2c^2$$

b)

(i)



By Newton's Second Law, resultant upward force on P has magnitude ma. Hence

$$ma = -\tfrac{1}{10}mv^2 - mg$$

$$a = -\left(\tfrac{1}{10}v^2 + 10\right) = -\left(\frac{v^2+100}{10}\right)$$

(ii)

$$\frac{dv}{dt} = -\left(\frac{v^2+100}{10}\right)$$

$$\frac{dt}{dv} = -\frac{10}{v^2+100}$$

$$t = -\tan^{-1}\left(\frac{v}{10}\right) + c$$

$$t=0, \ v=U \Rightarrow c = \tan^{-1}\left(\frac{U}{10}\right)$$

$$\therefore \ t = \tan^{-1}\left(\frac{U}{10}\right) - \tan^{-1}\left(\frac{v}{10}\right)$$

At maximum height, $v=0$ hence time to maximum height is $\tan^{-1}\left(\tfrac{1}{10}U\right)$ seconds.

(iii)

$$\tfrac{1}{2}\frac{dv^2}{dx} = -\left(\frac{v^2+100}{10}\right)$$

$$-\tfrac{1}{5}\frac{dx}{d(v^2)} = \frac{1}{(v^2)+100}$$

$$-\tfrac{1}{5}x = \ln(v^2+100)A, \quad A \text{ constant}$$

$$t=0, \ x=0, \ v=U \Rightarrow (U^2+100)A = 1$$

$$\therefore \quad -\tfrac{1}{5}x = \ln\left(\frac{v^2+100}{U^2+100}\right)$$

$$x = 5\ln\left(\frac{U^2+100}{v^2+100}\right)$$

At maximum height $v=0$. Hence maximum height is $5\ln\left(\frac{U^2+100}{100}\right)$ metres.

<u>QUESTION 15</u>

a)

Diameter of circular slice at height y is $x_2 - x_1 = 7y^{\frac{1}{4}} - y^{\frac{1}{4}} = 6y^{\frac{1}{4}}$. Hence slice at height y has area of cross section $\pi\left(3y^{\frac{1}{4}}\right)^2 = 9\pi y^{\frac{1}{2}}$, and volume $\delta V = 9\pi y^{\frac{1}{2}}\delta y$ where the thickness of the slice is δy.

Hence $V = \lim_{\delta y \to 0} \sum_{y=0}^{y=16} 9\pi y^{\frac{1}{2}}\delta y$ $\quad \therefore \quad V = 9\pi\int_0^{16} y^{\frac{1}{2}}\,dy = 6\pi\left[y^{\frac{3}{2}}\right]_0^{16} = 384\pi$

Volume is 384π cm^3 and capacity is $1\cdot 21$ litres (to 3 sig. fig.)

b)

Define the sequence of statements $S(n)$, $n=1,2,3,\ldots$ by $S(n)$: $T_n = \left(\sqrt{2}\right)^{n+2}\cos\frac{n\pi}{4}$

Consider $S(1)$, $S(2)$: $\left(\sqrt{2}\right)^{1+2}\cos\frac{1.\pi}{4} = 2\sqrt{2}\,.\,\frac{1}{\sqrt{2}} = 2 = T_1$ $\quad\therefore S(1)$ is true

$\left(\sqrt{2}\right)^{2+2}\cos\frac{2\pi}{4} = 4 \times 0 = 0 = T_2$ $\quad\therefore S(2)$ is true

If $S(n)$ is true, $n \le k$: $\quad T_n = \left(\sqrt{2}\right)^{n+2}\cos\frac{n\pi}{4}$, $\; n = 1,2,3,\ldots,k$ $\quad$ **

Consider $S(k+1)$, $k \ge 2$: $\quad T_{k+1} = 2T_k - 2T_{k-1}$ $\quad$ (since $k+1 \ge 3$)

$$= 2\left(\sqrt{2}\right)^{k+2}\cos\tfrac{k\pi}{4} - 2\left(\sqrt{2}\right)^{(k-1)+2}\cos\tfrac{(k-1)\pi}{4}, \quad \text{if } S(n) \text{ is true, } n \le k$$

$$= \left(\sqrt{2}\right)^{k+3}\left\{\sqrt{2}\cos\tfrac{k\pi}{4} - \cos\left(\tfrac{k\pi}{4} - \tfrac{\pi}{4}\right)\right\}$$

$$= \left(\sqrt{2}\right)^{k+3}\left\{2\,\tfrac{1}{\sqrt{2}}\cos\tfrac{k\pi}{4} - \left(\cos\tfrac{k\pi}{4}\cos\tfrac{\pi}{4} + \sin\tfrac{k\pi}{4}\sin\tfrac{\pi}{4}\right)\right\}$$

$$= \left(\sqrt{2}\right)^{k+3}\left\{2\,\tfrac{1}{\sqrt{2}}\cos\tfrac{k\pi}{4} - \tfrac{1}{\sqrt{2}}\cos\tfrac{k\pi}{4} - \tfrac{1}{\sqrt{2}}\sin\tfrac{k\pi}{4}\right)\right\}$$

$$= \left(\sqrt{2}\right)^{k+3}\left\{\tfrac{1}{\sqrt{2}}\cos\tfrac{k\pi}{4} - \tfrac{1}{\sqrt{2}}\sin\tfrac{k\pi}{4}\right\}$$

$$= \left(\sqrt{2}\right)^{k+3}\left\{\cos\tfrac{k\pi}{4}\cos\tfrac{\pi}{4} - \sin\tfrac{k\pi}{4}\sin\tfrac{\pi}{4}\right\}$$

$$= \left(\sqrt{2}\right)^{k+3}\cos\left(\tfrac{k\pi}{4} + \tfrac{\pi}{4}\right)$$

$$= \left(\sqrt{2}\right)^{(k+1)+2}\cos\tfrac{(k+1)\pi}{4}$$

$\therefore$ if $k \ge 2$ and $S(n)$ is true for $n \le k$, then $S(k+1)$ is true. But $S(1)$ and $S(2)$ are true, and hence $S(3)$ is true, and then $S(4)$ is true, and so on. Hence by Mathematical induction, $S(n)$ is true for all positive

c)

i. Vertical line through S has equation $x = ae$

At P, Q : $\quad \dfrac{(ae)^2}{a^2} - \dfrac{y^2}{b^2} = 1$

$$y^2 = b^2\left(e^2 - 1\right)$$

$$= \frac{b^4}{a^2}$$

Hence P, Q have coordinates $\left(ae\,,\pm\frac{b^2}{a}\right)$

$\therefore\; PQ = \dfrac{2b^2}{a}$

ii. $PQ = 48 \Rightarrow b^2 = 24a$

Also $\quad \dfrac{9^2}{a^2} - \dfrac{24^2}{b^2} = 1$

$$\frac{9^2}{a^2} - \frac{24}{a} = 1$$

$$a^2 + 24a - 81 = 0$$

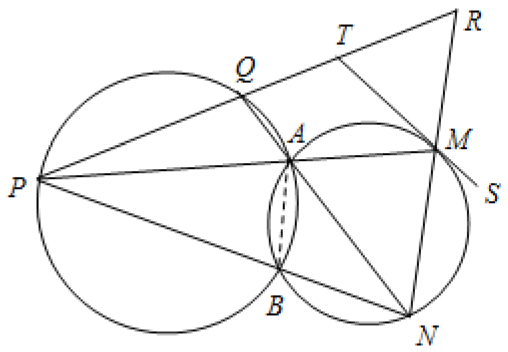
$$(a+27)(a-3) = 0$$

$$\therefore 0 < a < b \Rightarrow \begin{cases} a = 3 \\ b = 6\sqrt{2} \end{cases}$$

QUESTION 16

a)

i.



$\angle RMA = \angle ABN$ *(exterior angle of cyclic quad. ABNM is equal to interior opposite angle)*

Similarly
$\angle ABN = \angle AQP$ in cyclic quadrilateral $ABPQ$.

Hence quadrilateral $QAMR$ is cyclic.
(exterior angle AQP is equal to interior opposite angle RMA)

ii. Produce TM to S. Then
$\angle TMR = \angle SMN$ *(vertically opposite angles are equal)*
$\angle SMN = \angle MAN$ *(angle between tangent and chord drawn to point of contact is equal to angle subtended by that chord in the alternate segment)*
$\angle MAN = \angle PAQ$ *(vertically opposite angles are equal)*
$\angle PAQ = \angle TRM$ *(exterior angle of cyclic quad. QAMR is equal to interior opposite angle)*
Hence in ΔTMR, $\angle TMR = \angle TRM$ and hence $TM = TR$ *(sides opposite equal angles are equal)*

b)

$P(x) = x^4 - 2x^3 + 3x^2 - 4x + 1$. $\alpha,\ \beta,\ \gamma$ and δ are roots of $P(x) = 0$

i. Only possible integer roots are ± 1. But $P(1) = -1 \neq 0$ and $P(-1) = 11 \neq 0$. Hence there are no integer roots.

ii. $P(x)$ is a continuous, real function and $P(0) = 1 > 0$ while $P(1) = -1 < 0$. Hence, considering the graph of $y = P(x)$, there is a real root of $P(x) = 0$ between 0 and 1.

iii. $\alpha^2 + \beta^2 + \gamma^2 + \delta^2 = (\alpha + \beta + \gamma + \delta)^2 - 2(\alpha\beta + \alpha\gamma + \alpha\delta + \beta\gamma + \beta\delta + \gamma\delta) = 2^2 - 2 \times 3 = -2$

iv. Since $\alpha^2 + \beta^2 + \gamma^2 + \delta^2 = -2$, at least one of these squares must be negative. Hence $P(x) = 0$ has a non-real root. Then its complex conjugate is a second non-real root, since the coefficients of $P(x)$ are real. We know there is a real root between 0 and 1. Since the non-real roots come in complex conjugate pairs, the remaining fourth root cannot be non-real.
Hence the equation $P(x) = 0$ has two real roots and two non-real roots.

c)

$P(x) = x^3 - 6x^2 + 9x + c$
$P'(x) = 3x^2 - 12x + 9$
$= 3(x-3)(x-1)$
$\therefore P'(x) = 0$ for $x = 3$ or $x = 1$

$\therefore P'(3) = P(3) = 0 \iff 27 - 54 + 27 + c = 0 \iff c = 0$
and $P'(1) = P(1) = 0 \iff 1 - 6 + 9 + c = 0 \iff c = -4$
$\therefore P(x)$ has a double zero if and only if $c = 0$ or $c = -4$.