NORTH SYDNEY
GIRLS HIGH SCHOOL

2007
TRIAL HIGHER SCHOOL
CERTIFICATE

Mathematics Extension 2

Student Number:____________________________**Teacher:** ______________

Student Name:____________________________

General Instructions

- Reading time – 5 minutes.
- Working time – 3 hours.
- Write using black or blue pen.
- Board approved calculators may be used.
- All *necessary* working should be shown in every question if full marks are to be awarded.
- Marks may **NOT** be awarded for untidy or badly arranged work.
- Start each **NEW** question in a separate answer booklet.

Total Marks - 120 Marks

- Attempt Questions 1-8
- All questions are of equal value.

At the end of the examination, place your solution booklets in order and put this question paper on top.
Submit one bundle.
The bundle will be separated before marking commences so that anonymity will be maintained.

Question	1	2	3	4	5	6	7	8	Total
Mark									/120

Total marks – 120
Attempt Questions 1 - 8
All questions are of equal value

Answer each section in a SEPARATE writing booklet. Extra writing booklets are available.

Question 1 (15 marks) Use a SEPARATE writing booklet. **Marks**

(a) Evaluate $\displaystyle\int_0^2 \frac{x}{\sqrt{4+x^2}}\,dx$ **2**

(b) Using the substitution $u = e^x$, find $\displaystyle\int \frac{e^x}{\sqrt{1-e^{2x}}}\,dx$ **2**

(c) (i) Given that $\dfrac{5x^2-5x+14}{(x^2+4)(x-2)}$ can be written as **3**

$$\frac{5x^2-5x+14}{(x^2+4)(x-2)} = \frac{ax+b}{x^2+4} + \frac{c}{x-2}$$

where a, b and c are real numbers, find a, b and c.

(ii) Hence find $\displaystyle\int \frac{5x^2-5x+14}{(x^2+4)(x-2)}\,dx$ **2**

(d) Use the technique of *integration by parts* to evaluate **3**

$$\int_{-\frac{1}{2}}^{\frac{1}{2}} x\tan^{-1}2x\,dx$$

(e) Using the substitution $x = 2\sec\theta$ find **3**

$$\int \frac{1}{x\sqrt{x^2-4}}\,dx$$

Question 2 (15 marks) Use a SEPARATE writing booklet. **Marks**

(a) Let $z = 2 - i$ and $w = 3 - 2i$. Find, in the form $x + iy$,

(i) $(\bar{z})^2$ **1**

(ii) iwz **1**

(b) Given $z = 1 - \sqrt{3}\,i$, show that z^2 is a *real* multiple of $\frac{1}{z}$ **3**

(c) Sketch the region represented by **3**

$$|z| < 4 \text{ and } \frac{\pi}{3} < \arg z \leq \frac{2\pi}{3}$$

(d) (i) Show that $\dfrac{(1+i)^8}{\left(1-\sqrt{3}i\right)^k} = 2^{4-k}\left[\cos\left(\dfrac{k\pi}{3}\right) + i\sin\left(\dfrac{k\pi}{3}\right)\right]$ **3**

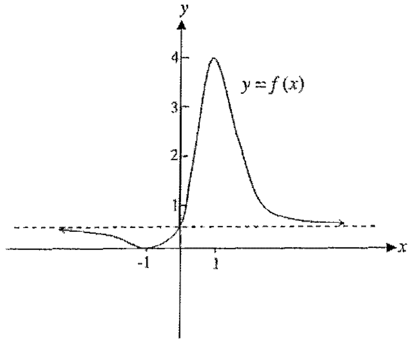
(ii) For what values of k is $\dfrac{(1+i)^8}{\left(1-\sqrt{3}i\right)^k}$ purely imaginary? **2**

(e) The equation $|z-3| - |z+3| = 4$ corresponds to a branch of a hyperbola in the Argand diagram. **2**

Sketch the branch, showing the length of the semi-transverse axis.

Question 3 (15 marks) Use a SEPARATE writing booklet. **Marks**

(a)



The graph of $y = f(x)$ is drawn above.
There are x- and y- intercepts at -1 and $y = 0\cdot5$ respectively. Also there are maximum and minimum turning points at respectively $(1, 4)$ and $(-1, 0)$.
The graph has a horizontal asymptote at $y = 0\cdot5$
On the Answer sheet provided sketch the following:

(i) $y = f(-x)$ **1**

(ii) $y = \dfrac{1}{f(x+1)}$ **2**

(iii) $y^2 = f(x)$ **2**

(iv) $y = \tan^{-1} f(x)$ **2**

(b) The equation $4x^2 + 9y^2 = 36$ is an ellipse. Find

(i) the x- and y- intercepts; **2**

(ii) its eccentricity; **1**

(iii) the coordinates of its foci; **1**

(iv) the equations of its directrices and then sketch its graph. **2**

(c) The following statements are either true or false. Write TRUE or FALSE for each statement and a brief reason for your answer. You are NOT required to evaluate the integrals.

(i) $\displaystyle\int_{-\frac{\pi}{4}}^{\frac{\pi}{4}} \cos 2x \sin 3x \, dx = 0$ **1**

(ii) $\displaystyle\int_0^1 \frac{dx}{\sqrt{1+x^3}} > \int_0^1 \frac{dx}{\sqrt{1+x^4}}$ **1**

Question 4 (15 marks) Use a SEPARATE writing booklet. **Marks**

(a) $1+i$ and $3-i$ are zeros of a real, monic polynomial, $p(x)$, of degree 4.

(i) Express $p(x)$ as a product of two real quadratic factors. **2**

(ii) Explain briefly why the polynomial $p(x)$ cannot take negative values for real values of x. **2**

(b) (i) Find the point of intersection, in the first quadrant, of the two graphs below **2**

$$x^2+10y^2=10$$
$$x^2-8y^2=8$$

(ii) Two graphs are said to be *orthogonal* if the product of their respective gradients at each point of intersection is -1. **2**
Show that the two graphs above are orthogonal at the point of intersection found in (i) above.

(c) (i) Given the hyperbola defined by $x=ct, y=\frac{c}{t}$, show that the equation of the tangent at the point where $t=p$ is $x+p^2y=2cp$. **2**

(ii) Show that the tangents, at the points p and q, meet at the point **2**

$$T\left(\frac{2cpq}{p+q},\frac{2c}{p+q}\right)$$

(iii) Find the equation of the locus of T if:

(α) $p+q=k$, where k is a constant and ignoring any restrictions on the domain. **1**

(β) $pq=K$, where K is a constant. **2**

Question 5 (15 marks) Use a SEPARATE writing booklet. **Marks**

(a) The region bounded by the curve $y=\cos^{-1}x$ and the x-axis, in the first quadrant, is rotated about the line $y=-1$. **4**

Using the method of cylindrical shells, find its volume.

(b) If $x=\frac{\pi}{4}-u$,

(i) Show that $\tan x=\frac{1-\tan u}{1+\tan u}$ **1**

(ii) Hence, or otherwise, show that $\int_0^{\frac{\pi}{4}}\ln(1+\tan x)\,dx=\frac{\pi}{8}\ln 2$ **5**

(c) Six lines are drawn in a plane. No two lines are parallel, and no three of the lines are concurrent.

(i) Show that there are 15 points of intersection. **1**

(ii) If three of these points are chosen at random, show that the probability that they all lie on one of the given lines is $\frac{12}{91}$. **2**

(iii) Find the probability that if four of these points are chosen at random they do not all lie on one of the given lines. **2**

Question 6 (15 marks) Use a SEPARATE writing booklet. **Marks**

(a) Figure 1 below shows a scale model of the volcano Mt Rekrap.
The base of the model is elliptical in shape with the axes 60 cm by 40 cm reducing uniformly to a circle of radius 12 cm at the top.
The hollow core of the model has circular cross sections with a circle of radius 6 cm at the base rising uniformly to a circle also of radius 12 cm at the top.
The model is 24 cm high.
Figure 2 shows the top view of the cross sectional area of the volcano.

Figure 1

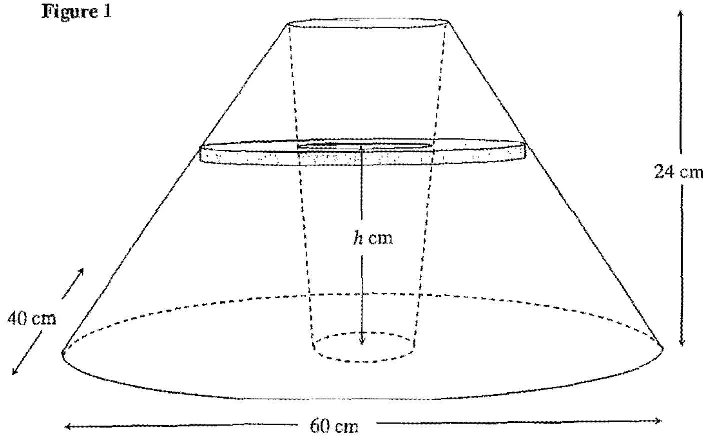
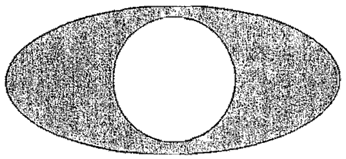


Figure 2



(i) Show that at height h, the length of the *semi-major axis* is given by **2**

$$a = 30 - \tfrac{3}{4}h$$

(ii) Show that the area of the cross sectional slice at height h is given by **4**

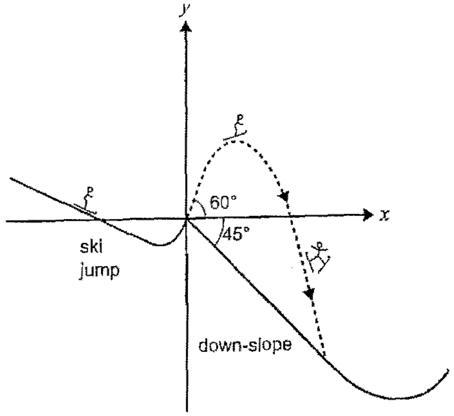
$$A = \frac{\pi}{16}\left(9024 - 448h + 3h^2\right)$$

You may assume that the area of an ellipse with semi-major axis a and semi-minor axis b is given by πab.

(iii) Find the volume of the scale model of Mt Rekrap. **2**

Question 6 continues on the next page

(b) A skier accelerates down a slope and then skis up a short ski jump as shown in the diagram below.
The skier leaves the jump at a speed of 12m/s and an angle of 60° to the horizontal, performs various gymnastic twists and lands on a straight line section of the 45° down-slope T seconds after leaving the jump.
Let the origin O of a Cartesian coordinate system be at the point where the skier leaves the jump.



(i) Assuming that $\ddot{x} = 0$ and $\ddot{y} = -g$, where g is the acceleration due to gravity, show that for $0 \le t \le T$ **3**

$$x = 6t$$
$$y = 6t\sqrt{3} - \tfrac{1}{2}gt^2$$

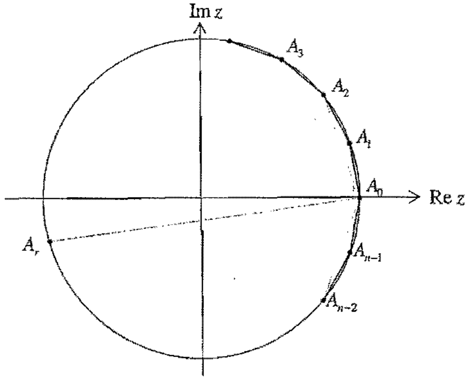
(ii) Show that $T = \dfrac{12}{g}\left(\sqrt{3}+1\right)$ **2**

(iii) At what speed does the skier land on the down-slope? **2**
Give your answer correct to one decimal place.

Question 7 (15 marks) Use a SEPARATE writing booklet. **Marks**

(a) The equation $x^3 - 8x^2 + 7 = 0$ has roots α, β and γ. **2**
Find a polynomial equation that has roots α^{-1}, β^{-1} and γ^{-1}.

(b) The Argand diagram below shows a regular n-sided polygon, with vertices $A_0, A_1, A_2, \dots, A_r, \dots, A_{n-1}$, which is inscribed in a unit circle with centre at $z = 0$.
A_0 lies on the positive real axis and corresponds to the number $z = 1$.
The other vertices are in anti-clockwise around the circle.
Let d_r be the length of the vector $\overrightarrow{A_0A_r}$ where $r = 1, 2, \dots, n-1$ and let P be defined by $P = d_1 d_2 \dots d_{n-1}$.
Also, let $\omega = \cos\left(\frac{2\pi}{n}\right) + i\sin\left(\frac{2\pi}{n}\right)$.



(i) Write down the complex numbers that correspond to the vertices $A_1, A_2, \dots, A_r, \dots, A_{n-1}$. Leave your answers in terms of ω **1**

(ii) By considering $z^n - 1 = 0$ deduce that **2**
$$(z-\omega)(z-\omega^2)\dots(z-\omega^{n-1}) = 1 + z + z^2 + \dots + z^{n-1}$$

(iii) Using the fact that $d_r = |1 - \omega^r|$, show that $P = n$ **2**

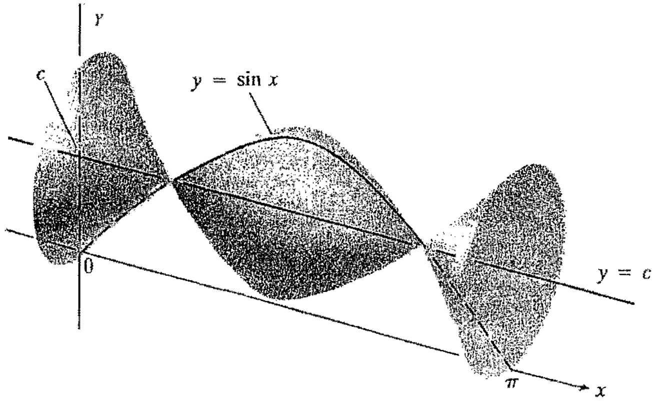
(iv) Using the fact that $z\bar{z} = |z|^2$ and $\cos 2\theta = 1 - 2\sin^2\theta$, show that **3**
$$|1 - \omega^r| = 2\sin\left(\frac{r\pi}{n}\right)$$

(v) Hence, or otherwise, find an expression, in terms of n, for **2**
$$\sin\left(\frac{\pi}{n}\right)\sin\left(\frac{2\pi}{n}\right)\dots\sin\left[\frac{(n-1)\pi}{n}\right]$$

(vi) Hence show that $\sin\left(\frac{\pi}{5}\right)\sin\left(\frac{2\pi}{5}\right) = \frac{\sqrt{5}}{4}$ **3**

Question 8 (15 marks) Use a SEPARATE writing booklet. **Marks**

(a) The arch $y = \sin x$, $0 \le x \le \pi$ is revolved around the line $y = c$ to generate the solid shown.



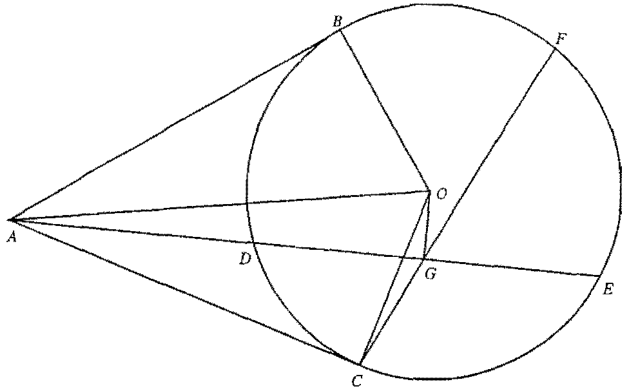
(i) Show that the volume of the arch can be represented by $\pi\int_0^{\pi}(\sin x - c)^2\,dx$ **2**

(ii) Find the value of c that minimises the volume. **3**

Question 8 continues over the page

Question 8 (continued) **Marks**

(b) In the diagram below, AB and AC are tangents from A to the circle with centre O, meeting the circle at B and C.
ADE is a secant of the circle. G is the midpoint of DE.
CG produced meets the circle at F.



(i) Show that A, O, G and C are concyclic points **2**

(ii) Explain why $\angle OGF = \angle OAC$ **1**

(iii) Show that $BF \parallel AE$ **2**

(c) (i) If $I_m = \int_0^{\varepsilon} x^m e^{-x}\,dx$ for $m > 0$, show that $I_m = mI_{m-1} - e^{-\varepsilon}\varepsilon^m$ for $m \geq 1$ **3**

(ii) If $J_m = \lim_{\varepsilon \to \infty} I_m$, show that $J_m = mJ_{m-1}$ for $m \geq 1$ **1**

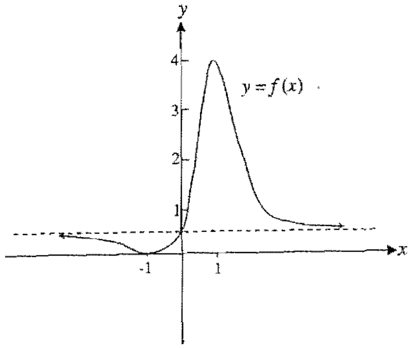
(iii) Deduce that $J_m = m!$ for $m \geq 1$ **1**

End of paper

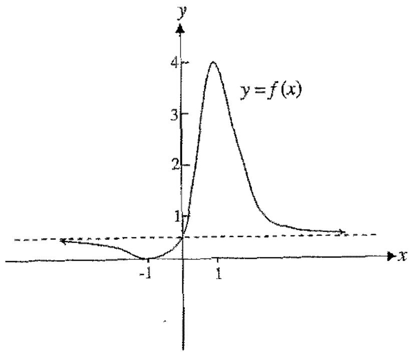
NSGHS 2007 Trial HSC Extension 2 Mathematics Exam Answer Sheet for Question 3(a)

Number____________________ Teacher:________

(i)

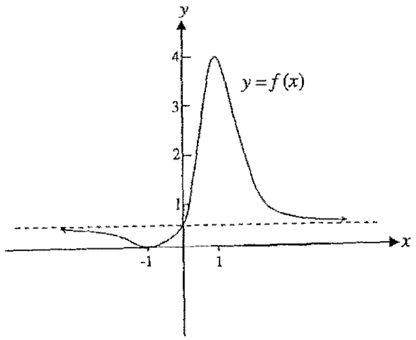


(ii)

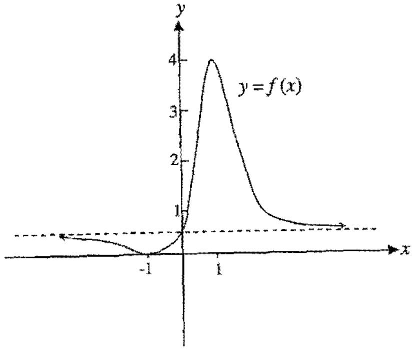


Turn over for parts (iii) and (iv)

(iii)



(iv)



PLACE THIS SHEET INSIDE YOUR BOOKLET FOR QUESTION 3

Question 1

(a) $\int_0^2 \frac{x}{\sqrt{4+x^2}}\,dx = \frac{1}{2}\int_0^2 \frac{2x}{\sqrt{4+x^2}}\,dx$

$\left[\begin{array}{l} u = 4+x^2 \Rightarrow du = 2x\,dx \\ x = 0, u = 4; x = 2, u = 8 \end{array}\right]$

$= \frac{1}{2}\int_4^8 \frac{du}{\sqrt{u}} = \frac{1}{2}\int_4^8 u^{-\frac{1}{2}}\,du$

$= \left[u^{\frac{1}{2}}\right]_4^8 = 2\sqrt{2} - 2$

(b) $u = e^x \Rightarrow du = e^x dx$

$\int \frac{e^x}{\sqrt{1-e^{2x}}}\,dx = \int \frac{du}{\sqrt{1-u^2}}$

$= \sin^{-1}(u) + c$

$= \sin^{-1}(e^x) + c$

(c) (i) $\frac{5x^2 - 5x + 14}{(x^2+4)(x-2)} = \frac{ax+b}{x^2+4} + \frac{c}{x-2}$

$\therefore 5x^2 - 5x + 14 = (x-2)(ax+b) + c(x^2+4)$

Substitute $x = 2$: $\quad 24 = \quad c(8)$

$\therefore c = 3$

$a + c = 5 \quad$ (coefficient of x^2)

$\therefore a = 2$

Substitute $x = 0$: $\quad 14 = \quad (-2)b + 4c$

$14 = -2b + 12$

$\therefore b = -1$

$\therefore a = 2, b = -1, c = 3$

(ii) $\int \frac{5x^2 - 5x + 14}{(x^2+4)(x-2)}\,dx = \int \left(\frac{2x-1}{x^2+4} + \frac{3}{x-2}\right)dx$

$= \int \frac{2x}{x^2+4}\,dx - \int \frac{1}{x^2+4}\,dx + \int \frac{3}{x-2}\,dx$

$= \ln(x^2+4) - \frac{1}{2}\tan^{-1}\left(\frac{x}{2}\right) + 3\ln|x-2| + k$

(d) $\int_{-\frac{1}{2}}^{\frac{1}{2}} x\tan^{-1}2x\,dx = 2\int_0^{\frac{1}{2}} \frac{d\left(\frac{1}{2}x^2\right)}{dx}\tan^{-1}2x\,dx$ [even function]

$$= 2\left[\frac{1}{2}x^2\tan^{-1}2x\right]_0^{\frac{1}{2}} - 2\int_0^{\frac{1}{2}}\left(\frac{1}{2}x^2 \times \frac{2}{1+4x^2}\right)dx$$

$$= \frac{1}{4}\times\frac{\pi}{4} - \int_0^{\frac{1}{2}}\frac{2x^2}{1+4x^2}dx = \frac{\pi}{16} - \frac{1}{2}\int_0^{\frac{1}{2}}\frac{(4x^2+1)-1}{1+4x^2}dx$$

$$= \frac{\pi}{16} - \frac{1}{2}\int_0^{\frac{1}{2}} 1\,dx + \frac{1}{2}\int_0^{\frac{1}{2}}\frac{1}{1+4x^2}dx$$

$$= \frac{\pi}{16} - \frac{1}{2}\times\frac{1}{2} + \frac{1}{4}\int_0^{\frac{1}{2}}\frac{2}{1+4x^2}dx$$

$$= \frac{\pi}{16} - \frac{1}{4} + \frac{1}{4}\left[\tan^{-1}2x\right]_0^{\frac{1}{2}}$$

$$= \frac{\pi}{16} - \frac{1}{4} + \frac{1}{4}\times\frac{\pi}{4}$$

$$= \frac{\pi}{8} - \frac{1}{4}$$

(e) $x = 2\sec\theta \Rightarrow dx = 2\sec\theta\tan\theta\,d\theta$

$x^2 - 4 = 4\sec^2\theta - 4 = 4(\sec^2\theta - 1) = 4\tan^2\theta$

$$\int \frac{1}{x\sqrt{x^2-4}}dx = \int \frac{1}{2\sec\theta\sqrt{4\tan^2\theta}}2\sec\theta\tan\theta\,d\theta$$

$$= \int \frac{1}{2}d\theta$$

$$= \frac{\theta}{2} + c$$

$$= \frac{1}{2}\sec^{-1}\left(\frac{x}{2}\right) + c \qquad \left[\sec\theta = \frac{x}{2} \Rightarrow \theta = \sec^{-1}\left(\frac{x}{2}\right)\right]$$

$$= \frac{1}{2}\cos^{-1}\left(\frac{2}{x}\right) + c$$

Question 2

(a) (i) $\left(\overline{2-i}\right)^2 = (2+i)^2 = 4 - 1 + 4i = 3 + 4i$

(ii) $i(2-i)(3-2i) = (1+2i)(3-2i) = 3 + 4 + i(6-2) = 7 + 4i$

(b) $z = 1 - \sqrt{3}i = 2\text{cis}\left(-\frac{\pi}{3}\right)$

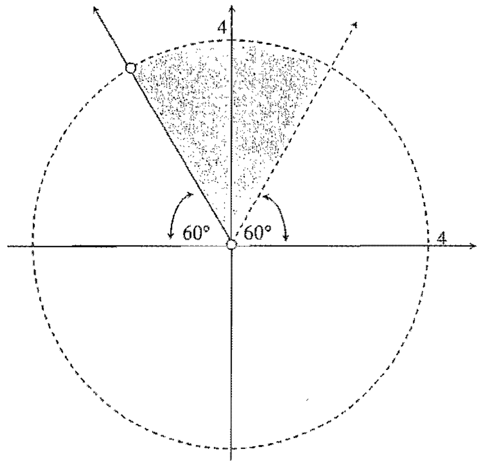
$\therefore z^2 = 4\text{cis}\left(-\frac{2\pi}{3}\right)$ [de Moivre's Th^m]

also $\frac{1}{z} = z^{-1} = \frac{1}{2}\text{cis}\left(\frac{\pi}{3}\right)$

$$\frac{z^2}{z^{-1}} = \frac{4\text{cis}\left(-\frac{2\pi}{3}\right)}{\frac{1}{2}\text{cis}\left(\frac{\pi}{3}\right)} = 8\text{cis}(-\pi) = -8$$

$\therefore z^2 = -8 \times \frac{1}{z}$

(c)



(d) (i) $1+i=\sqrt{2}\text{cis}\left(\frac{\pi}{4}\right), 1-\sqrt{3}i=2\text{cis}\left(-\frac{\pi}{3}\right)$

$$\frac{(1+i)^8}{\left(1-\sqrt{3}i\right)^k}=\frac{\left[\sqrt{2}\text{cis}\left(\frac{\pi}{4}\right)\right]^8}{\left[2\text{cis}\left(-\frac{\pi}{3}\right)\right]^k}=\frac{2^4\text{cis}2\pi}{2^k\text{cis}\left(-\frac{k\pi}{3}\right)} \quad [\text{de Moivre's Th}^{\text{m}}]$$

$$=2^{4-k}\times\frac{\text{cis}0}{\text{cis}\left(-\frac{k\pi}{3}\right)}=2^{4-k}\text{cis}\left(\frac{k\pi}{3}\right)$$

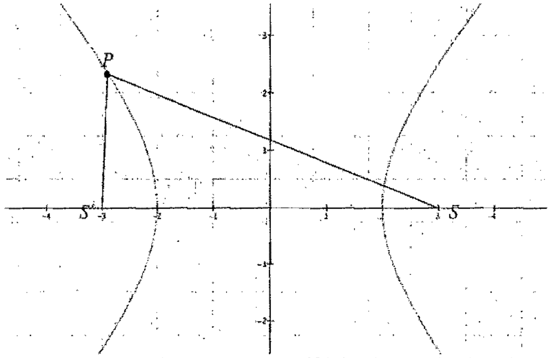
$$=2^{4-k}\left[\cos\left(\frac{k\pi}{3}\right)+i\sin\left(\frac{k\pi}{3}\right)\right]$$

(ii) $\frac{(1+i)^8}{\left(1-\sqrt{3}i\right)^k}$ is purely imaginary if $\text{Re}\left[\frac{(1+i)^8}{\left(1-\sqrt{3}i\right)^k}\right]=0$

$$\therefore \cos\left(\frac{k\pi}{3}\right)=0 \Rightarrow \frac{k\pi}{3}=2n\pi\pm\frac{\pi}{2}, n\in\mathbb{Z}$$

$$\therefore k=6n\pm\frac{3}{2}=\frac{3}{2}(4n\pm1)$$

(e) The foci of the hyperbola are $S(3,0)$ and $S'(-3,0)$, with a transverse axis, $2a$, of 4.
So the length of the semi-transverse axis is given by $a=2$.
The equation says that $SP-S'P=4\ (>0)$, where P is any point on the hyperbola

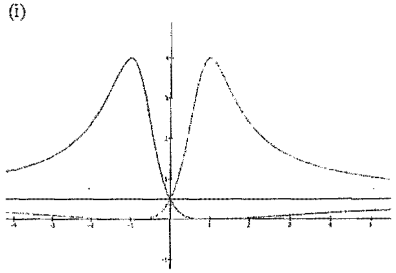


So the branch must be the left one, as if P is taken on the right then $SP-S'P<0$

Question 3

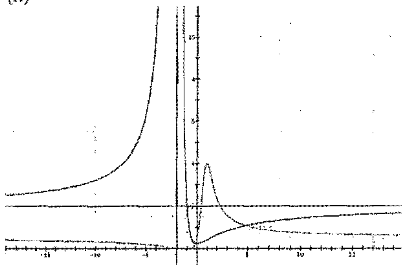
(a)

(i)



$y=f(-x)$ is a reflection in the y-axis

(ii)



First, there is a lateral shift to the left of one unit.
The maximum is at $(0,4)$ and the minimum is at $(-2,0)$.

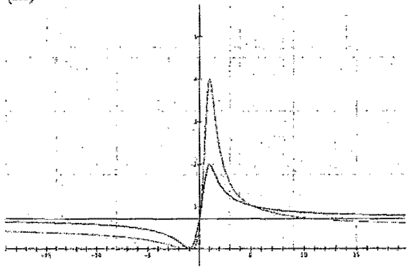
With the reciprocal there will be:

(*) a vertical asymptote at $x=-2$

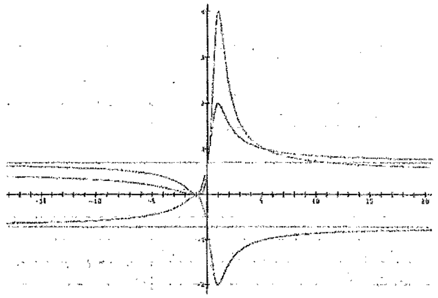
(*) a minimum at $\left(0,\frac{1}{4}\right)$

(*) a horizontal asymptote at $y=2$

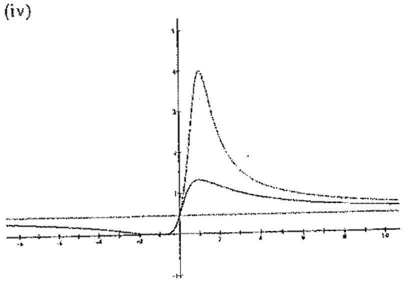
(iii)



First looking at $y=\sqrt{f(x)}$:
We have $f(x)\geq 0$.
The maximum is at $(1,2)$ and a horizontal asymptote at $y=\frac{1}{\sqrt{2}}$.
At $x=-1$, there is a double root.
Also when $f(x)>1$, then $f(x)>\sqrt{f(x)}$
and when $f(x)<1$, then $f(x)<\sqrt{f(x)}$
To draw $y^2=f(x)$, reflect in the x-axis.

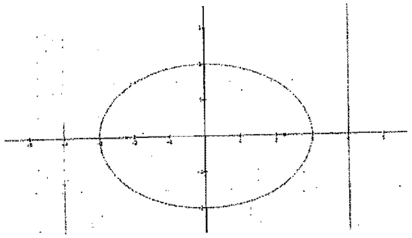


(iv)



The minimum is at $\left(1, \tan^{-1} 4\right) \approx (1, 1\cdot 3)$.
The horizontal asymptote is at
$y = \tan^{-1}\left(\frac{1}{2}\right) \approx 0\cdot 46$

(b) $4x^2 + 9y^2 = 36 \Rightarrow \frac{x^2}{9} + \frac{y^2}{4} = 1$



(i) $(\pm 3, 0), (0, \pm 2)$

(ii) $a^2 = 9, b^2 = 4$

$e^2 = 1 - \frac{b^2}{a^2} = 1 - \frac{4}{9} = \frac{5}{9}$

$\therefore e = \frac{\sqrt{5}}{3}$

(iii) Foci $(\pm ae, 0) = (\pm\sqrt{5}, 0)$

(iv) $x = \pm\frac{a}{e} = \pm\frac{9}{\sqrt{5}} = \pm\frac{9\sqrt{5}}{5}$

(c) (i) $\int_{-\frac{\pi}{4}}^{\frac{\pi}{4}} \cos 2x \sin 3x \, dx = 0$ **TRUE** since $\cos 2x \sin 3x$ is an ODD function.

(ii) $0 \le x \le 1 \Rightarrow x^3 > x^4$

$\therefore 1 + x^3 > 1 + x^4 \Rightarrow \sqrt{1+x^3} > \sqrt{1+x^4}$

$\therefore \frac{1}{\sqrt{1+x^3}} < \frac{1}{\sqrt{1+x^4}} \Rightarrow \int_0^1 \frac{dx}{\sqrt{1+x^3}} < \int_0^1 \frac{dx}{\sqrt{1+x^4}}$

$\therefore$ **FALSE**

Question 4

(a) (i) $p(1+i) = 0 \Rightarrow p(1-i) = 0$ [Conjugate root Th^m]

Similarly $p(3+i) = 0$

$p(x) = [x-(1+i)][x-(1-i)][x-(3-i)][x-(3+i)]$

$= (x^2 - 2x + 2)(x^2 - 6x + 10)$

$\left[(x-\alpha)(x-\bar{\alpha}) = x^2 - 2(\mathrm{Re}\,\alpha)x + |\alpha|^2\right]$

(ii) $p(x) = \left[(x^2 - 2x + 1) + 1\right]\left[(x^2 - 6x + 9) + 1\right]$

$= \left[(x-1)^2 + 1\right]\left[(x-3)^2 + 1\right]$

Now $(x-1)^2 + 1 > 0$ for $x \in \mathbb{R}$. Similarly, $(x-3)^2 + 1 > 0$ for $x \in \mathbb{R}$

$\therefore p(x) = \left[(x-1)^2 + 1\right]\left[(x-3)^2 + 1\right] > 0$ for $x \in \mathbb{R}$

(b) (i) $x^2 + 10y^2 = 10 \quad -(1)$

$x^2 - 8y^2 = 8 \quad -(2)$

$(1) - (2)$: $18y^2 = 2$

$\therefore y^2 = \frac{1}{9} \Rightarrow y = \frac{1}{3}$

$\therefore x^2 + \frac{10}{9} = 10 \Rightarrow x^2 = \frac{80}{9}$

$\therefore x = \frac{4\sqrt{5}}{3}$

The point of intersection is $\left(\frac{4\sqrt{5}}{3}, \frac{1}{3}\right)$

(ii) Differentiate (1): $2x + 20y.y' = 0$

$y' = -\frac{x}{10y} = -\frac{2\sqrt{5}}{5}$ at $\left(\frac{4\sqrt{5}}{3}, \frac{1}{3}\right)$

Differentiate (2): $2x - 16y.y' = 0$

$y' = \frac{x}{8y} = \frac{\sqrt{5}}{2}$ at $\left(\frac{4\sqrt{5}}{3}, \frac{1}{3}\right)$

Now $-\frac{2\sqrt{5}}{5} \times \frac{\sqrt{5}}{2} = -1$, so the two curves are ORTHOGONAL.

(c) (i) $\frac{dx}{dt}=c, \frac{dy}{dt}=-\frac{c}{t^2}$

$\therefore \frac{dy}{dx}=\frac{dy}{dt}\div\frac{dx}{dt}=-\frac{1}{t^2}$

$\therefore m=-\frac{1}{p^2}$ when $t=p$

$\therefore y-\frac{c}{p}=-\frac{1}{p^2}(x-cp)$

$\therefore p^2y-cp=-x+cp$

$\therefore\therefore y-\frac{c}{p}=-\frac{1}{p^2}(x-cp)$

$\therefore p^2y-cp=-x+cp$

$\therefore x+p^2y=2cp$

(ii) Similarly $x+q^2y=2cq$ is the tangent when $t=q$.

$x+p^2y=2cp \quad -(1)$

$x+q^2y=2cq \quad -(2)$

$(1)-(2)$: $(p^2-q^2)y=2c(p-q)$

$\therefore y=\frac{2c(p-q)}{(p^2-q^2)}=\frac{2c(p-q)}{(p-q)(p+q)}=\frac{2c}{p+q} \quad [p\neq q]$

Substitute into (1):

$x+p^2\left(\frac{2c}{p+q}\right)=2cp \Rightarrow (p+q)x+2cp^2=2cp(p+q)$

$\therefore (p+q)x=2cpq \Rightarrow x=\frac{2cpq}{p+q}$

$\therefore T\left(\frac{2cpq}{p+q},\frac{2c}{p+q}\right)$

(iii) (α) If $p+q=k \Rightarrow T\left(\frac{2cpq}{p+q},\frac{2c}{k}\right)$

$\therefore y=\frac{2c}{k}$

[NB $x=\frac{2cp(k-p)}{k}=\frac{2c}{k}\times p(k-p)$

$p(k-p)$ is a quadratic expression that has a maximum of $x=\frac{2c}{k}\times\frac{k^2}{4}=\frac{ck}{2}$ when $p=\frac{k}{2}$.

So the locus of T is $y=\frac{2c}{k}$ for $x\leq\frac{ck}{2}$]

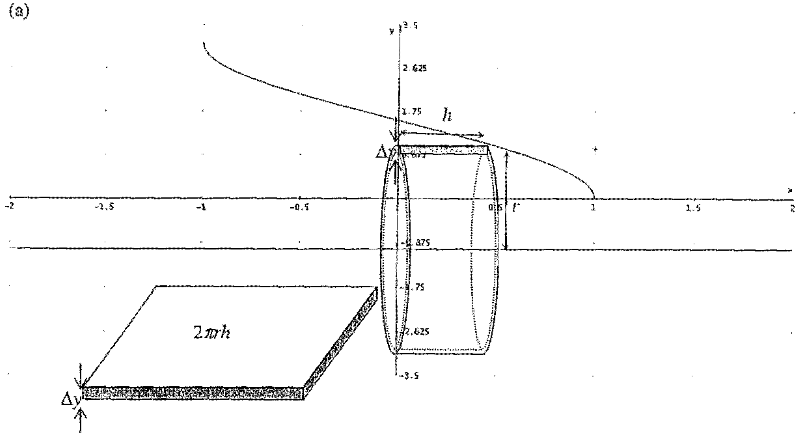
(β) If $pq=K \Rightarrow T\left(\frac{2cpq}{p+q},\frac{2c}{p+q}\right)$

$\therefore \frac{y}{x}=\frac{2c}{2cpq}=\frac{1}{pq}=\frac{1}{K}$

$\therefore y=\frac{x}{K}$, BUT $pq\neq 0 \Rightarrow (0, 0)$ is excluded

Question 5

(a)



$h=x=\cos y$

$r=y+1$

$\Delta V \approx 2\pi rh\Delta y=2\pi(y+1)\cos y\Delta y$

$V=\int_0^{\frac{\pi}{2}}2\pi(y+1)\cos y\, dy$

$=2\pi\int_0^{\frac{\pi}{2}}(y+1)\frac{d}{dy}(\sin y)dy$

$=2\pi[(y+1)\sin y]_0^{\frac{\pi}{2}}-2\pi\int_0^{\frac{\pi}{2}}\sin y\, dy \qquad \left[\frac{d}{dy}(y+1)\right]=1$

$=2\pi\left(\frac{\pi}{2}+1\right)-2\pi[-\cos y]_0^{\frac{\pi}{2}}$

$=\pi^2$

(b) (i) $\tan x=\tan\left(\frac{\pi}{4}-u\right)$

$=\frac{\tan\frac{\pi}{4}-\tan u}{1+\tan\frac{\pi}{4}\tan u}$

$=\frac{1-\tan u}{1+\tan u}$

(ii) $u=\frac{\pi}{4}-x;\ du=-dx$

$x=0, u=\frac{\pi}{4};\ x=\frac{\pi}{4}, u=\frac{\pi}{4}$

$$\ln(1+\tan x)=\ln\left(1+\frac{1-\tan u}{1+\tan u}\right)=\ln\left(\frac{2}{1+\tan u}\right)=\ln 2-\ln(1+\tan u)$$

$$\int_0^{\frac{\pi}{4}}\ln(1+\tan x)dx=\int_{\frac{\pi}{4}}^{0}[\ln 2-\ln(1+\tan u)](-du)$$

$$=\int_0^{\frac{\pi}{4}}(\ln 2)du-\int_0^{\frac{\pi}{4}}\ln(1+\tan u)du$$

$$2\int_0^{\frac{\pi}{4}}\ln(1+\tan x)dx=\ln 2\left(\frac{\pi}{4}-0\right)=\frac{\pi\ln 2}{4}$$

$$\int_0^{\frac{\pi}{4}}\ln(1+\tan x)dx=\frac{\pi\ln 2}{8}$$

NB $\int_0^{\frac{\pi}{4}}\ln(1+\tan u)du=\int_0^{\frac{\pi}{4}}\ln(1+\tan x)dx$

(c) (i) With six lines, an intersection is obtained by choosing any two lines ie $\binom{6}{2}=15$

(ii) Any line has 5 points of intersection with the remaining lines. Choosing 3 of these points can be done in $\binom{5}{3}=10$ ways.

Taking all six lines, there are $6\times 10=60$ ways to have three points on the same line.

There are $\binom{15}{3}=455$ ways of choosing any three points from all the intersections.

So the probability is $\frac{60}{455}=\frac{12}{91}$.

ALTERNATIVE 1:

The probability of picking any point is 1. Then this point lies on two lines, so the probability of picking another point that lies on both lines is $\frac{8}{14}$ as there are 4 remaining points on both lines. Once this point is picked, the line is defined and so picking the last point is $\frac{3}{13}$. Giving the probability as $1\times\frac{8}{14}\times\frac{3}{13}=\frac{12}{91}$.

For (iii), this will give $1\times\frac{8}{14}\times\frac{3}{13}\times\frac{2}{12}=\frac{2}{91}$ and then take complement.

ALTERNATIVE 2:

The probability of picking a point on a given line is $\frac{5}{15}$. To pick another point on the line is $\frac{4}{14}$ and finally $\frac{3}{13}$ to pick the last oint. As there are 6 lines, the probability is $\frac{5}{15}\times\frac{4}{14}\times\frac{3}{13}\times 6=\frac{12}{91}$. For (iii) use $\frac{5}{15}\times\frac{4}{14}\times\frac{3}{13}\times\frac{2}{12}\times 6=\frac{2}{91}$ and then take complement.

(iii) First find the probability that the 4 points all lie on the line above.

Using the method above this would be $\frac{6\times\binom{5}{4}}{\binom{15}{4}}=\frac{30}{1365}=\frac{2}{91}$

So the probability that they don't all lie on the same line is the probability of its COMPLEMENTARY event ie $1-\frac{2}{91}=\frac{89}{91}$

Question 6

(a) (i) From the diagram $2a=60$ ie $a=30$ when $h=0$ and when $h=24$, $a=$ radius $=12$

h	0	24
a	30	12

By similar triangles this relationship is linear.

$$\therefore a=\left(\frac{12-30}{24-0}\right)h+30=30-\frac{3}{4}h$$

(ii) Similarly for b and the radius R of the internal circle.

h	0	24
b	20	12

$$\therefore b=\left(\frac{12-20}{24-0}\right)h+20=20-\frac{1}{3}h$$

h	0	24
R	6	12

$$\therefore R=\left(\frac{12-6}{24-0}\right)h+6=6+\frac{1}{4}h$$

So the area, A, of the cross sectional slice is given by $A=\pi ab-\pi R^2=\pi\left(ab-R^2\right)$

$$A=\pi\left[\left(20-\frac{1}{3}h\right)\left(30-\frac{3}{4}h\right)-\left(6+\frac{1}{4}h\right)^2\right]$$

$$=\pi\left[\left(\frac{60-h}{3}\right)\left(\frac{120-3h}{4}\right)-\left(\frac{24+h}{4}\right)^2\right]$$

$$=\frac{\pi}{16}\left[4(60-h)(40-h)-(24+h)^2\right]$$

$$=\frac{\pi}{16}\left[4\left(2400-100h+h^2\right)-\left(576+48h+h^2\right)\right]$$

$$=\frac{\pi}{16}\left(9024-448h+3h^2\right)$$

(iii) So if the cross sectional slice has a thickness, Δh, then the volume of the slice is

$$\Delta V\approx\frac{\pi}{16}\left(9024+448h+3h^2\right)\Delta h$$

and so the volume, V, of Mt Rekrap is given by

$$V=\frac{\pi}{16}\int_0^{24}\left(9024-448h+3h^2\right)dh$$

$$=\frac{\pi}{16}\left[9024h-224h^2+h^3\right]_0^{24}$$

$$=\frac{\pi}{16}(216576-129024+13824)$$

$$=\frac{\pi}{16}\times 101376$$

$$=6336\pi$$

(b) $t=0, x=0, y=0$

$t=0, \dot{x}=12\cos 60^\circ = 6, \dot{y}=12\sin 60^\circ = 6\sqrt{3}$

(i)

$\ddot{x}=0$
$\therefore \dot{x}=k$ (constant)
$[t=0, \dot{x}=6]$
$\therefore \dot{x}=6$
$\therefore x=6t+k_1$
$[t=0, x=0]$
$\therefore x=6t$

$\ddot{y}=-g$
$\therefore \dot{y}=-gt+c$ (constant)
$[t=0, \dot{y}=6\sqrt{3}]$
$\therefore \dot{y}=-gt+6\sqrt{3}$
$\therefore y=-\frac{1}{2}gt^2+6t\sqrt{3}+c_2$
$[t=0, y=0]$
$\therefore y=-\frac{1}{2}gt^2+6t\sqrt{3}$

(ii) At $t=T$, the skier lands at a point where $y=-x$

$\therefore -\frac{g}{2}T^2+6T\sqrt{3}=-6T$

$\therefore T\left(-\frac{g}{2}T+6\sqrt{3}+6\right)=0$

$\therefore -\frac{g}{2}T+6\sqrt{3}+6=0 \quad (T\neq 0)$

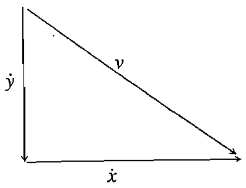
$\therefore \frac{g}{2}T=6\sqrt{3}+6$

$\therefore T=\frac{12}{g}\left(\sqrt{3}+1\right)$

(iii) At $t=T, \dot{x}=6$

$\dot{y}=-gT+6\sqrt{3}=-12\left(\sqrt{3}+1\right)+6\sqrt{3}=-6\left(\sqrt{3}+2\right)$

$v^2=(\dot{x})^2+(\dot{y})^2$
$=6^2+\left[-6\left(\sqrt{3}+2\right)\right]^2$



$\therefore v\approx 23\cdot 2$

The skier lands with a speed of $23\cdot 2$ m/s (correct to 1 decimal place)
NB the skier does not land at an angle of 45°.

Question 7

(a) Let $y=\frac{1}{x}\Rightarrow x=\frac{1}{y}$

$\therefore \left(\frac{1}{y}\right)^3-8\left(\frac{1}{y}\right)^2+7=0$

$\therefore \frac{1}{y^3}-\frac{8}{y^2}+7=0$

$\therefore 1-8y+7y^3=0$

(b) (i) $A_1\leftrightarrow \omega, A_2\leftrightarrow \omega^2,\ldots, A_r\leftrightarrow \omega^r,\ldots, A_{n-1}\leftrightarrow \omega^{n-1}$

(ii) $z^n-1=0\Rightarrow z=1,\omega,\omega^2,\ldots,\omega^{n-1}$

$\therefore z^n-1$ can be factorised in two ways:

$z^n-1=(z-1)(z-\omega)(z-\omega^2)\ldots(z-\omega^{n-1})$
$=(z-1)(1+z+z^2+\ldots+z^{n-1})$

$\therefore (z-\omega)(z-\omega^2)\ldots(z-\omega^{n-1})=(1+z+z^2+\ldots+z^{n-1})$

(iii) $P=d_1d_2\ldots d_{n-1}=|1-\omega|\times|1-\omega^2|\times\ldots\times|1-\omega^{n-1}|$

$=|(1-\omega)(1-\omega^2)\ldots(1-\omega^{n-1})|$ $\quad[|zw|=|z|\times|w|]$

$=|1+1+1^2+\ldots+1^{n-1}|$ $\quad$[Let $z=1$ and using (ii)]

$=|n|$

$=n$

(iv) $|1-\omega^r|^2=\left|1-\text{cis}\left(\frac{2\pi r}{n}\right)\right|^2=\left|\left(1-\cos\left(\frac{2\pi r}{n}\right)\right)-i\sin\left(\frac{2\pi r}{n}\right)\right|^2$

$=\left[\left(1-\cos\left(\frac{2\pi r}{n}\right)\right)-i\sin\left(\frac{2\pi r}{n}\right)\right]\times\left[\left(1-\cos\left(\frac{2\pi r}{n}\right)\right)+i\sin\left(\frac{2\pi r}{n}\right)\right]$ $\quad[z\bar{z}=|z|^2]$

$=\left(1-\cos\left(\frac{2\pi r}{n}\right)\right)^2+\sin^2\left(\frac{2\pi r}{n}\right)$

$=1-2\cos\left(\frac{2\pi r}{n}\right)+\cos^2\left(\frac{2\pi r}{n}\right)+\sin^2\left(\frac{2\pi r}{n}\right)$

$=2-2\cos\left(\frac{2\pi r}{n}\right)=2\left(1-\cos\left(\frac{2\pi r}{n}\right)\right)$

$=2\left(2\sin\left(\frac{\pi r}{n}\right)\right)$ $\quad[1-\cos 2\theta=2\sin^2\theta]$

$=4\sin^2\left(\frac{\pi r}{n}\right)$

$\therefore |1-\omega^r|^2=4\sin^2\left(\frac{\pi r}{n}\right)$

$\therefore |1-\omega^r|=2\sin\left(\frac{\pi r}{n}\right)$ $\quad[|z|>0]$

(v) $$2\sin\left(\frac{\pi}{n}\right)\times 2\sin\left(\frac{2\pi}{n}\right)\times\ldots\times 2\sin\left[\frac{(n-1)\pi}{n}\right]=|1-\omega|\times|1-\omega^2|\times\ldots\times|1-\omega^{n-1}| \quad \text{[From (iv)]}$$
$$= d_1 d_2 \ldots d_{n-1}$$
$$= P$$
$$= n \quad \text{[From (iii)]}$$

$$\therefore 2^{n-1}\left[\sin\left(\frac{\pi}{n}\right)\sin\left(\frac{2\pi}{n}\right)\ldots\sin\left[\frac{(n-1)\pi}{n}\right]\right]=n$$
$$\therefore \sin\left(\frac{\pi}{n}\right)\sin\left(\frac{2\pi}{n}\right)\ldots\sin\left[\frac{(n-1)\pi}{n}\right]=\frac{n}{2^{n-1}}$$

(vi) $n=5$

$$\therefore \sin\left(\frac{\pi}{5}\right)\sin\left(\frac{2\pi}{5}\right)\sin\left(\frac{3\pi}{5}\right)\sin\left(\frac{4\pi}{5}\right)=\frac{5}{2^4}=\frac{5}{16} \quad \text{[From (v)]}$$

NB $\sin\left(\frac{2\pi}{5}\right)=\sin\left(\frac{3\pi}{5}\right)$; $\sin\left(\frac{\pi}{5}\right)=\sin\left(\frac{4\pi}{5}\right)$

$$\therefore \sin^2\left(\frac{\pi}{5}\right)\sin^2\left(\frac{2\pi}{5}\right)=\frac{5}{16}$$

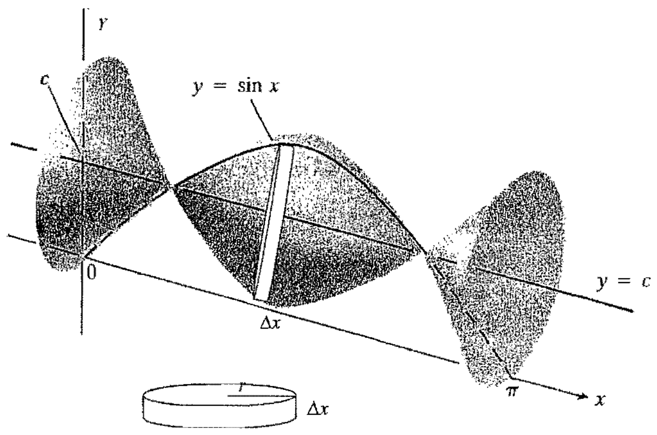
$$\therefore \sin\left(\frac{\pi}{5}\right)\sin\left(\frac{2\pi}{5}\right)=\sqrt{\frac{5}{16}}=\pm\frac{\sqrt{5}}{4}$$

$$\left[\begin{array}{l} 0<\frac{\pi}{5},\frac{2\pi}{5}<\frac{\pi}{2} \\ \therefore \sin\left(\frac{\pi}{5}\right)\sin\left(\frac{2\pi}{5}\right)>0 \end{array}\right]$$

$$\therefore \sin\left(\frac{\pi}{5}\right)\sin\left(\frac{2\pi}{5}\right)=\frac{\sqrt{5}}{4}$$

Question 8

(a) (i) Take a slice of thickness Δx and rotate it around $y=c$.
This will form an approximate cylindrical disk of radius r $(=|\sin x - c|)$ and height Δx



So the volume, ΔV, of this "disk" is given by:

$$\Delta V = \pi r^2 \Delta x$$
$$= \pi\left(|\sin x - c|^2\right)\Delta x$$
$$= \pi(\sin x - c)^2 \Delta x$$

So the volume, V, of the solid is given by

$$V=\pi\int_0^{\pi}(\sin x - c)^2\,dx$$

(ii) $V=\pi\int_0^{\pi}(\sin x-c)^2\,dx$

$$=\pi\int_0^{\pi}\sin^2 x\,dx-2\pi c\int_0^{\pi}\sin x\,dx+\pi\int c^2dx$$

$$=\frac{\pi}{2}\int_0^{\pi}2\sin^2 x\,dx+2\pi c\int_0^{\pi}-\sin x\,dx+\pi c^2\int_0^{\pi}1\,dx$$

$$=\frac{\pi}{2}\int(1-\cos 2x)dx+2\pi c\Big[\cos x\Big]_0^{\pi}+\pi c^2\times\pi$$

$$=\frac{\pi}{2}\left[x-\frac{\sin 2x}{2}\right]_0^{\pi}-4\pi c+\pi^2c^2$$

$$=\frac{\pi^2}{2}-4\pi c+\pi^2c^2$$

$$=\frac{\pi}{2}\left(\pi-8c+2\pi c^2\right)$$

$$V=\frac{\pi}{2}\left(\pi-8c+2\pi c^2\right)$$

$$V'=\frac{\pi}{2}(-8+4\pi c)$$

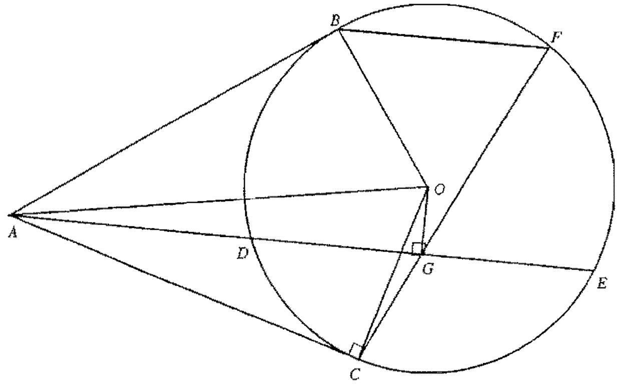
$$V''=\frac{\pi}{2}(4\pi)>0$$

Minimum V when $V'=0\Rightarrow -8+4\pi c=0$

$$\therefore c=\frac{8}{4\pi}=\frac{2}{\pi}$$

Given that $V''>0$, with $c=\frac{2}{\pi}$ then V is minimised.

(b) Construct BF



(i) $OG\perp DE$ (G midpoint of DE)

$\angle ACO=90°$ (AC tangent)

$\therefore\angle OGA=\angle ACO$

$\therefore$ A, O, G and C are concyclic points (converse of angles in the same segment)

(ii) $\angle OGF$ is the exterior angle to $\angle OAC$.

$\therefore\angle OAC=\angle OGF$ (exterior angle of a cyclic quadrilateral)

(iii) Let $\angle BFG=x$

$\therefore\angle BOC=2x$ (angles at the centre and circumference)

$\therefore\angle AOC=x$ (AO axis of symmetry in $AOGC$)

$\therefore\angle OAC=\frac{\pi}{2}-x$ (angle sum of right – angled Δ)

$\therefore\angle OGF=\frac{\pi}{2}-x$ (from (ii))

$\therefore\angle FGE=x$ ($OG\perp DE$)

$\therefore DE\parallel BF$ (alternate angles are equal)

(c) (i) $I_m = \int_0^{\varepsilon} x^m e^{-x} dx = \int_0^{\varepsilon} \frac{d}{dx}\left(-e^{-x}\right)x^m \, dx$

$$= \left[-e^{-x}x^m\right]_0^{\varepsilon} - \int mx^{m-1}\left(-e^{-x}\right)dx$$

$$= -\varepsilon^m e^{-\varepsilon} + m\int x^{m-1}e^{-x}dx$$

$$= mI_{m-1} - e^{-\varepsilon}\varepsilon^m$$

(ii) $J_m = \lim_{\varepsilon\to\infty}\left(mI_{m-1} - e^{-\varepsilon}\varepsilon^m\right)$

$$= mJ_{m-1} - \lim_{\varepsilon\to\infty}\left(e^{-\varepsilon}\varepsilon^m\right)$$

$$= mJ_{m-1} \qquad \left[\lim_{\varepsilon\to\infty}\left(e^{-\varepsilon}\varepsilon^m\right) = 0 \text{ as } e^x \text{ dominates } x^m\right]$$

(iii) $J_m = mJ_{m-1}$

$$= m\times(m-1)J_{m-2}$$

$$= m\times(m-1)\times(m-2)J_{m-3}$$

$$= m\times(m-1)\times(m-2)\times\ldots\times1\times J_0$$

$$J_0 = \lim_{\varepsilon\to\infty}\int_0^{\varepsilon} e^{-x}dx = \lim_{\varepsilon\to\infty}\left[-e^{-x}\right]_0^{\varepsilon} = \lim_{\varepsilon\to\infty}\left(-e^{-\varepsilon}+1\right) = 1$$

$$\therefore J_m = m(m-1)(m-2)\times\ldots\times1\times \underset{J_0}{1} = m!$$