MATHEMATICS

2014 HSC Course Assessment Task 3 (Trial Examination)
June 18, 2014

General Instructions

- Working time –3 hours
  (plus 5 minutes reading time).
- Write using blue or black pen.
  Diagrams may be sketched in pencil.
- Board approved calculators may be used.
- All necessary working should be shown in every question.
- Attempt all questions.

Section I - 10 marks

- Mark your answers on the answer sheet provided.

Section II – 90 marks

- Commence each new question on a new page.
- Show all necessary working in every question. Marks may be deducted for illegible or incomplete working.

STUDENT NUMBER: …………………………………………… **# BOOKLETS USED:** ……….

Class (please ✓)

☐ Mr Lin ☐ Mr Berry

☐ Mr Weiss ☐ Mr Lowe

☐ Mr Zuber

Question	1-10	11	12	13	14	15	16	**Total**
Marks	/10	/15	/15	/15	/15	/15	/15	/100

Section 1: Multiple Choice– 1 mark each.

Q1. $\dfrac{\pi + \sqrt{3}}{4 - 1.1^5}$ to 3 significant figures is:

(A) 1.964

(B) 1.96

(C) 2.039

(D) 2.04

Q2. The equation $2x^2 + bx + 6 = 0$ has a root at $x = 3$.

What is the value of b?

(A) -2

(B) -5

(C) -7

(D) -8

Q3. In a group of 50 students, 22 study Economics and 13 study Music. 6 of these students study both Economics and Music.

What is the probability a randomly selected student studies neither subject?

(A) 0.42

(B) 0.58

(C) 0.70

(D) 0.82

Q4. Which of the following describes the series:

$$\log_2 2\,, \log_2 4\,, \log_2 8\,, \ldots$$

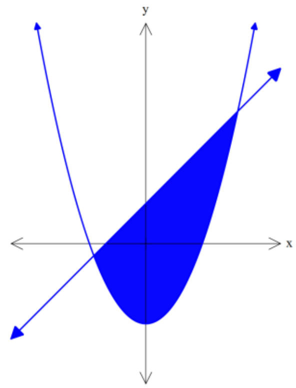
(A) An arithmetic series with common difference 2.

(B) An arithmetic series with common difference $\log_2 2$.

(C) A geometric series with common ratio 2.

(D) A geometric series with common ratio $\log_2 2$.

Q5. Which statement is consistent with the region shown?



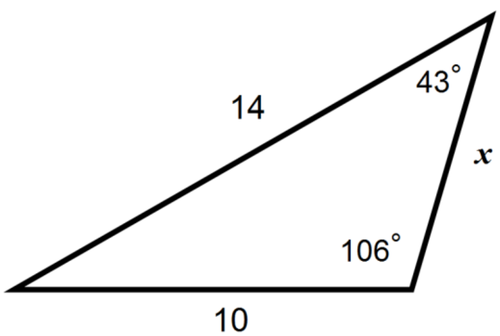
(A) $2y \geq x^2 - 8$ and $x - y + 2 \geq 0$

(B) $2y \leq x^2 - 8$ and $x - y + 2 \geq 0$

(C) $2y \geq x^2 - 8$ and $x - y + 2 \leq 0$

(D) $2y \leq x^2 - 8$ and $x - y + 2 \leq 0$

Q6. In the triangle below,



Which one of the following statements is true?

(A) $x = \sin 43° \cdot \dfrac{14}{\sin 106°}$

(B) $x = \sin 31° \cdot \dfrac{14}{\sin 43°}$

(C) $x = \sin 31° \cdot \dfrac{10}{\sin 43°}$

(D) $x = \sin 74° \cdot \dfrac{10}{\sin 43°}$

Q7. The solution to $3x^2 + 13x > 10$ is

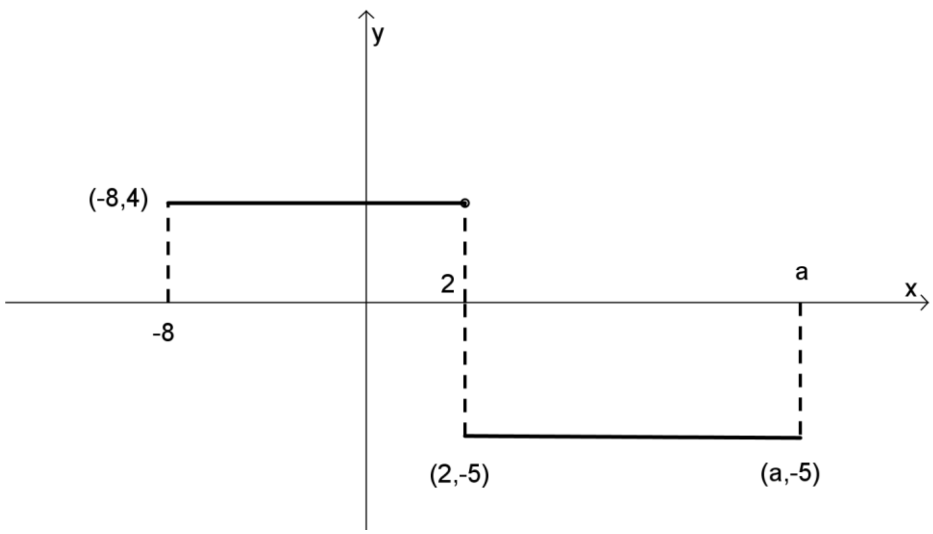
(A) $x < -\frac{2}{3}$ or $x > 5$

(B) $-5 < x < \frac{2}{3}$

(C) $x < -5$ or $x > \frac{2}{3}$

(D) $-\frac{2}{3} < x < 5$

Q8. Using the graph of $y = f(x)$ below,



NOT TO SCALE

determine the value of a which satisfies the condition:

$$\int_{-8}^{a} f(x)dx = 0$$

(A) 4

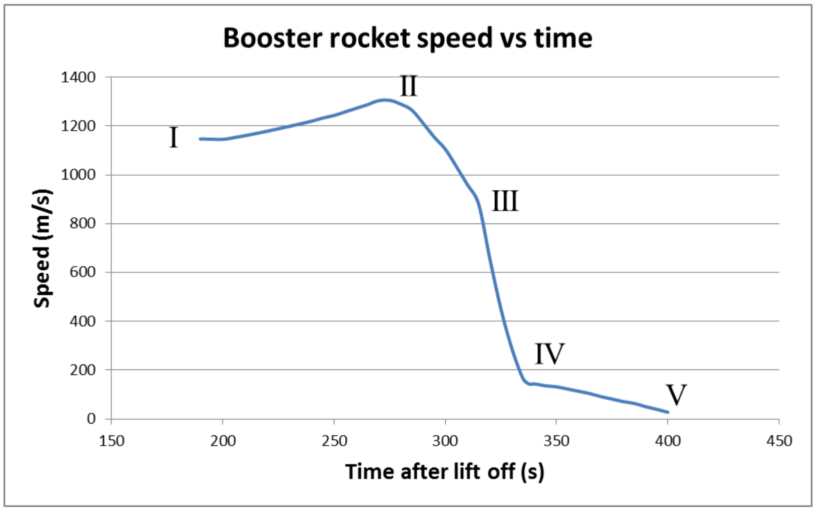
(B) 8

(C) 10

(D) 12

Q9. The NASA space shuttle was launched into orbit with the aid of two booster rockets. After the fuel in the booster rockets was depleted, they were released and allowed to fall back to Earth.

The following graph shows the speed of a booster rocket during its return journey to Earth.



Which one of the following statements can be reliably interpreted from the graph?

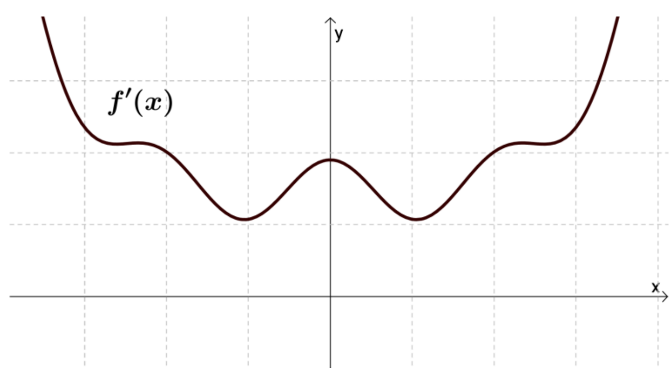
(A) The booster rocket began falling towards the Earth at point (II).

(B) The booster rocket experienced acceleration between points (II) and (III).

(C) The area under the curve provides the height the booster rocket was released.

(D) The booster rocket experienced maximum deceleration between points (III) and (IV).

Q10.



Looking at the graph of $y = f'(x)$ above, which of the following is true?

(A) $f(x)$ is an odd function with a local maximum at $f(0)$.

(B) $f(x)$ is an odd function with a point of inflexion at $f(0)$.

(C) $f(x)$ is an even function with a local maximum at $f(0)$.

(D) $f(x)$ is an even function with a point of inflexion at $f(0)$.

End of Section I

Section II – Short Answer 90 marks

Question 11 (15 marks) Commence on a NEW page — Marks

(a) Simplify $\dfrac{\sqrt{75}-\sqrt{3}}{2}$ — 2

(b) Find $\displaystyle\lim_{x\to 2}\frac{2x-4}{x^2-4}$ — 2

(c) Find the exact value of $\tan\theta$ given $\cos\theta = \frac{3}{7}$ and $\theta > 90°$ — 2

(d) What is the exact value of $\sec\frac{5\pi}{6}$? — 2

(e) Differentiate with respect to x : $y = 3x^3 - \frac{1}{x^2}$ — 2

(f) Find $\displaystyle\int (x^3 + \sqrt{x}\,)\,dx$ — 2

(g) Find and then graph the solution to $|\,4x + 2\,| < 5$ — 3

Question 12 (15 marks) Commence on a NEW page. — Marks

(a) The points $A(-2,-1)$, $B(5,1)$, $C(0,2)$ are defined in the Cartesian plane.

i) Find the equation of a line passing through points A and B — 2

i) Find the equation of the line parallel to AB passing through point C. — 2

ii) Hence, or otherwise, find the distance between the two parallel lines. — 1

(b) Differentiate with respect to x:

i) $\cos(x^3 - 2x)$ — 2

ii) $\dfrac{x}{1+e^x}$ — 2

(c) Find:

i) $\displaystyle\int 3e^{-5x}\,dx$ — 2

ii) $\displaystyle\int x^2\left(1-\sqrt{x}\right)dx$ — 2

iii) $\displaystyle\int_1^2 \frac{4x}{x^2+1}\,dx$ — 2

Question 13 (15 marks) Commence on a NEW page. Marks

(a) The equation of a parabola is given as 3

$$y^2 - 4y - 16x - 12 = 0$$

Sketch the parabola, clearly showing the location of the vertex, the focus point and the directrix.

(b) The brightness I of a distant star is observed to vary according to the formula

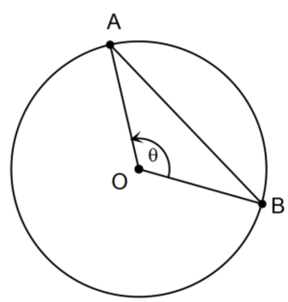
$$I = 53 + 3\sin\left(\frac{t}{4}\right)$$

where t is the time in hours.

i) What is the period of I? 1

ii) What is the range I? 1

(c) Points A and B are on the circumference of a circle centre O, radius 14 cm. The chord AB has length 22 cm.



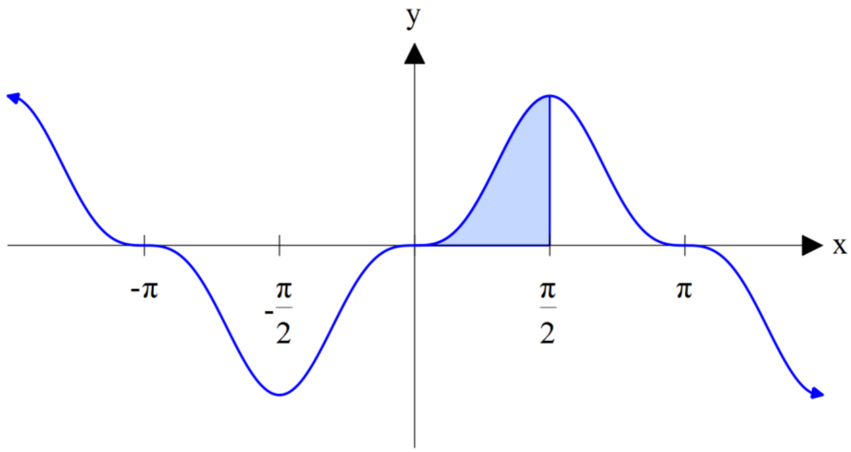
i) Find the size of the angle θ subtended by the minor arc AB (nearest degree). 2

ii) Hence find the area of minor segment defined by chord AB to 1 decimal place. 2

Question 13 continues on the next page.

Question 13 (continued)

(d) The following is the graph of $y = \sin^3 x$.



i) Use the Trapezoidal Rule with three values to estimate the area bounded by $y = \sin^3 x,\ x = 0,\ x = \frac{\pi}{2}$ and the x-axis to 2 decimal places. 3

ii) Based on your observation of the graph, what is the value of

$$\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \sin^3 x \ dx$$ 1

(e) Find the sum of the series 2

$$1 - \frac{2}{3} + \frac{4}{9} - \frac{8}{27} + \frac{16}{81} + \cdots$$

End of Question 13

Question 14 (15 marks) Commence on a NEW page. Marks

(a) Given the equation

$$y = -x(x-5)^2 = -x^3 + 10x^2 - 25x$$

i) Find any stationary points and determine their nature. 2

ii) Find any points of inflexion. 2

iii) Hence sketch the curve, clearly indicating the intercepts, stationary points and points of inflexion. 4

(b) The following three statements can be found in Australia Post's 2013 Annual Report:

"In 2008 our mail volumes peaked around 5.6 billion items per year."

"Since 2008, our mail volumes have steadily declined by around 4.5% per year."

"In 2012-13 we delivered one billion fewer letters than we did five years earlier"

i) Assuming an exponential decline in mail volume, 2

$$V(t) = A\,e^{-kt} \quad (t \text{ in years since 2008})$$

use the first two statements above to estimate appropriate values for A and k.

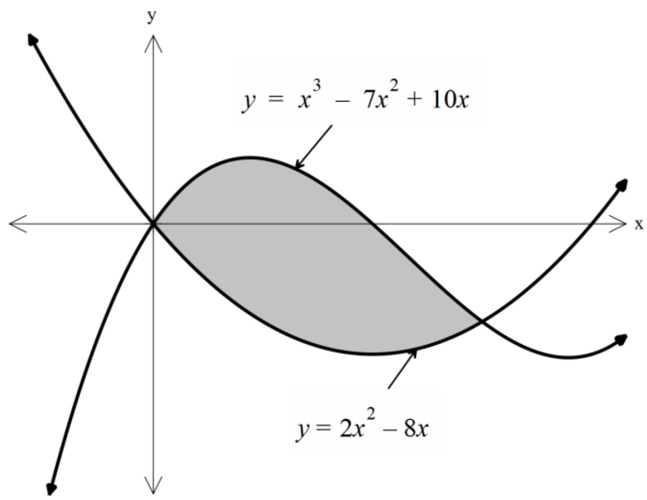
ii) Is the resulting exponential model consistent with the third statement? 1

Use a date of June 2013 for the time period "2012-2013".

Question 14 continues on the next page.

Question 14 (continued)

(c) The graphs $y = 2x^2 - 8x$ and $y = x^3 - 7x^2 + 10x$ are shown in the diagram below.



One intersection point is at $(0,0)$.

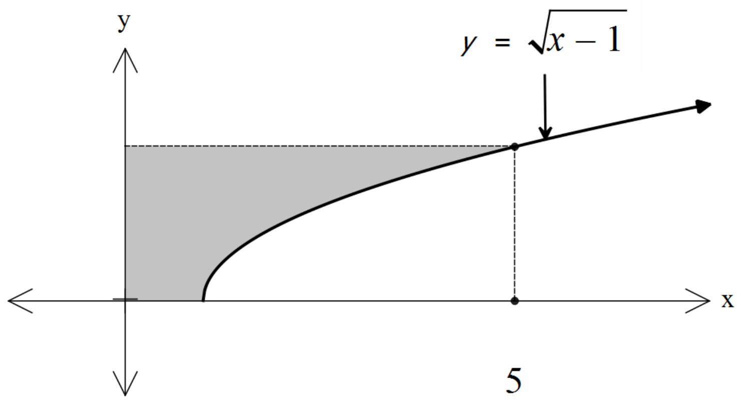
i) Show that another intersection point occurs at $(3, -6)$. 1

ii) Find the exact area between the two curves. 3

End of Question 14

Question 15 (15 marks) Commence on a NEW page. Marks

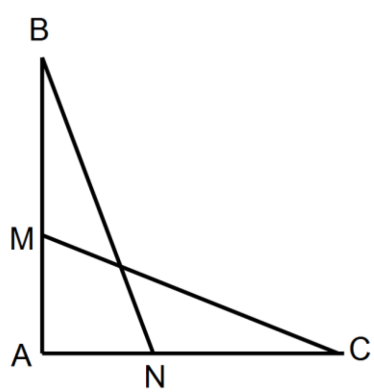
(a) The curve $y = \sqrt{x-1}$ in the domain $x = 1$ to $x = 5$ is rotated about the y-axis. 4



Find the volume of the resulting solid of revolution.

(b) Two equal length intervals AB and AC are drawn.

Point M and N are placed on the intervals with the condition $\angle ABN = \angle ACM$.



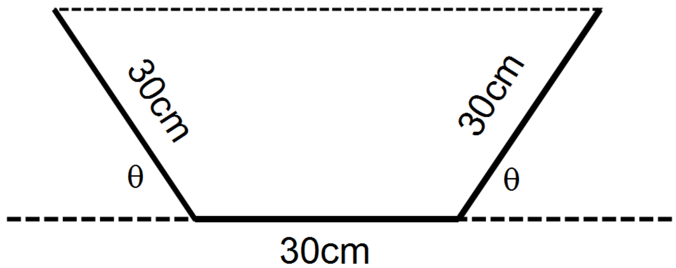
i) Prove that $\Delta ABN \equiv \Delta ACM$. 2

ii) Hence prove that $BM = CN$. 2

Question 15 continues on the next page.

(c) A flat sheet of metal 90 cm wide is folded upwards 30 cm from each edge to form the base for a watering trough.

The cross-section of the trough is in the shape of an isosceles trapezium:



i) Show that the cross sectional area can be expressed in the form: 3

$$A = 900(\sin\theta + \sin\theta\cos\theta)$$

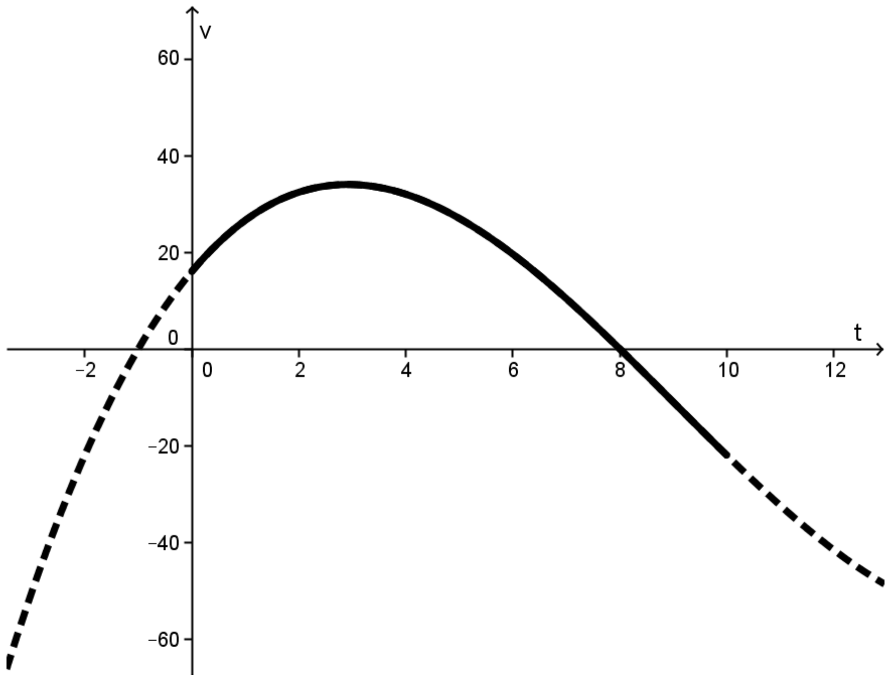
ii) Find the value of the angle θ which will maximise the cross sectional area. 4

End of Question 15

Question 16 (15 marks) Commence on a NEW page.

(a) A gazelle is grazing under a tree. At time $t = 0$ seconds, a cheetah, 100 m to the left of the tree, sees the gazelle and begins a chase.

The following is a graph of the velocity of the cheetah's pursuit:



The cheetah's velocity fits the equation:

$$v(t) = 0.1(t + 1)(t - 8)(t - 20) = 0.1(t^3 - 27t^2 + 132t + 160)$$

where t is the time in seconds in the range $0 \leq t \leq 10$.

The cheetah captures the gazelle time $t = 10$ seconds.

Assume the cheetah runs in a straight line.

i) What distance did the cheetah run during the 10 second chase? 3

ii) The graph shows the cheetah reached maximum velocity at some time between time t=2 and t=4 seconds. 2

At what time exactly did the cheetah reach this maximum velocity (to 1 DP)?

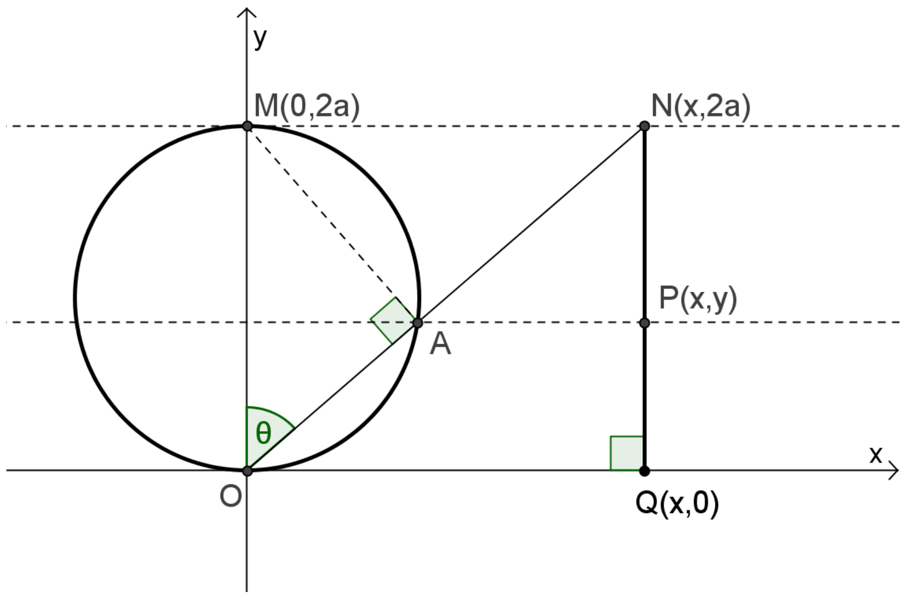
Question 16 continues on the next page

Question 16 (continued)

(b) A circle is drawn using the origin O and point $M(0, 2a)$ as the diameter.

A line segment of height $2a$ is drawn between points $Q(x, 0)$ and $N(x, 2a)$.

Points O and N are joined to form a line, creating an intersection point A on the circle. The point A is then projected on NQ to produce point $P(x, y)$



The angle θ is formed by OM and ON.

Given: $\angle OAM$ will always be a right angle, regardless of the location of A.

i) Show that the x coordinate of point P is given by **1**

$$x = 2a \tan \theta$$

ii) Show that the y coordinate of point P is given by **2**

$$y = 2a \cos^2 \theta$$

iii) Hence or otherwise show that the coordinates of the point P are given by the expression: **2**

$$y = \frac{8a^3}{x^2 + 4a^2}$$

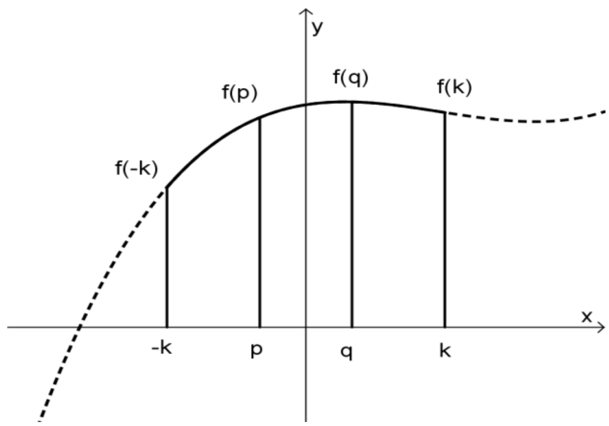
Question 16 continues on the next page.

Question 16 (continued)

c) The Trapezoidal Rule approximates an integral by fitting two points to a line. Simpson's Rule approximates an integral by fitting three equally spaced points a quadratic curve.

The following question explores the development of a rule using four equally spaced points fitted to a cubic curve.

Let $f(x) = Ax^3 + Bx^2 + Cx + D$, defined for $-k \leq x \leq k$, where A, B, C, D are constants.



i) Show that **1**

$$\int_{-k}^{k} f(x)\,dx = 2k\left(\frac{B}{3}k^2 + D\right)$$

ii) The interval $-k \leq x \leq k$ is divided into three equally sized intervals to produce four four points at $x = -k,\ x = p,\ x = q,\ x = k.$ **2**

Express p and q in terms of k and hence show that:

$$f(-k) + 3f(p) + 3f(q) + f(k) = 8\left(\frac{B}{3}k^2 + D\right)$$

iii) Hence write an expression that could be used to find **2**

$$\int_a^b f(x)\,dx$$

using only the value of a and b, and the four values $f(a),\ f\left(\frac{2a+b}{3}\right),\ f\left(\frac{a+2b}{3}\right),\ f(b)$.

End of paper.

Solutions 2U

Q1. $2.0396 \approx 2.04$ (3 sig figs) (D)

Q2. $2(3)^2 + b(3) + 6 = 0$ (D)

$\Rightarrow b = -8$

or $\alpha\beta = \frac{6}{2}$

$3\beta = 3$

$\beta = 1$

$(\alpha + \beta) = -\frac{b}{2}$

$-2(3+1) = b$

$b = -8$

Q3. $P(E) = \frac{22}{50}$ (A)

$P(M) = \frac{13}{50}$

$P(E \cap M) = \frac{6}{50}$

$P(E \cup M) = \frac{22+13-6}{50}$

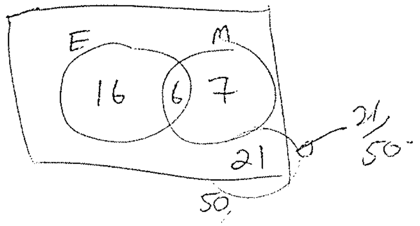
$= \frac{29}{50}$

$\overline{P(E \cup M)} = 1 - \frac{29}{50}$

$= \frac{21}{50}$

$= 0.42$

or



1. D
2. D
3. A
4. B.
5. A
6. C
7. C
8. C
9. D.
10. B

Q4. $\log_2 2, \log_2 4, \log_2 8$

is $\log_2 2, 2\log_2 2, 3\log_2 8$ (B)

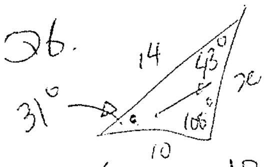
Arithmetic series, common difference $\log_2 2$

Q5. Test point $(0,0)$ (A)

$2(0) \geq (0)^2 - 8$ $\quad 2y \geq x^2 - 8$

$(0) - (0) + 2 \geq 0$ $\quad x - y + 2 \geq 0$

Q6.



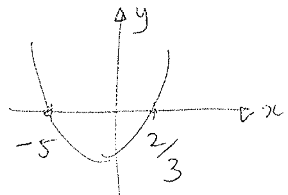
$\frac{x}{\sin 31°} = \frac{10}{\sin 43°}$ (C)

$x = \sin 31° \times \frac{10}{\sin 43°}$

Q7. $3x^2 + 13x > 10$ (C)

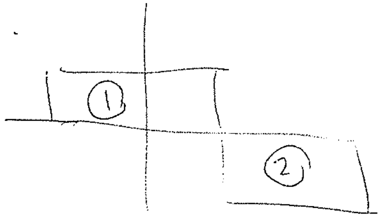
$3x^2 + 13x - 10 > 0$

$(3x-2)(x+5) > 0$



$x < -5$ or $x > 2/3$

Q8.



D

Area 1 $= 10 \times 4 = 40\,u^2$

Want Area 2 $= 40\,u^2$

$w \times 5 = 40$

$w = 8$

$a = 2 + w$

$= 10.$

Q9.

(A) False – graph is speed. Can't tell direction (re

(B) False – speed reducing – decelerating.

(C) False – may not be falling in straight line, don't know direction.

(D) True.

Q10. ~~Grow~~ $f(x)$ is always > 0, so always increasing.

$\to$ no turning points.

$f''(0) = 0$, $f'(x)$ not changing sign $\to$ inflexion

(B) Odd function with inflexion at $f(0)$

Q11.

a) $\dfrac{\sqrt{75} - \sqrt{3}}{2} = \dfrac{5\sqrt{3} - \sqrt{3}}{2}$ ✓

$= \dfrac{4\sqrt{3}}{2}$

$= 2\sqrt{3}$ ✓

b) $\displaystyle\lim_{x \to 2} \frac{2x-4}{x^2-4} = \lim_{x \to 2} \frac{2(x-2)}{(x-2)(x+2)}$ ✓

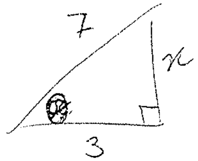
$= \displaystyle\lim_{x \to 2} \frac{2}{x+2}$

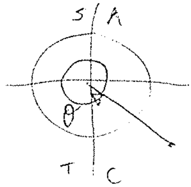
$= \dfrac{2}{2+2}$

$= \frac{1}{2}$ ✓

c) $\cos\theta = \frac{3}{7}$

$\theta > 90°$





Quadrant IV

$x^2 = 7^2 - 3^2$

$= 49 - 9$

$x = \sqrt{40}$

$= 2\sqrt{10}$

θ in quadrant IV $\Rightarrow \tan\theta < 0$

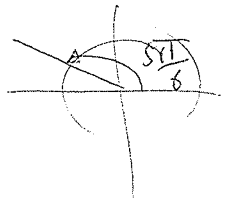
$\tan\theta = -\dfrac{2\sqrt{10}}{3}$

✓ progress

✓ answer.

(5)

1) $\sec \frac{5\pi}{6} = \left(\cos \frac{5\pi}{6}\right)^{-1}$



$= \left(-\cos\frac{\pi}{6}\right)^{-1}$ ✓

$= \left(-\frac{\sqrt{3}}{2}\right)$

$= \frac{-2}{\sqrt{3}}$ ✓

$y = 3x^3 - \frac{1}{x^2} = 3x^3 - x^{-2}$

$y' = 9x^2 + 2x^{-3}$ ✓ ✓ } 1 mark for each term

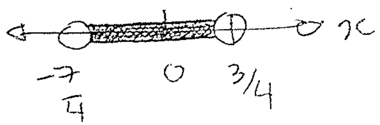
$\int (x^3 + \sqrt{x})\,dx = \frac{1}{4}x^4 + \frac{2}{3}x^{3/2} + C$ } ✓ ✓ mark for each term. Deduct 1 if no $+C$

) $|4x+2| < 5$

$-5 < 4x+2 < 5$ ✓ for progress

$-7 < 4x < 3$ ✓ for boundary values

$-\frac{7}{4} < x < \frac{3}{4}$ ✓ for graph with holes



(6)

P12

$A(-2,-1)$, $B(5,1)$, $C(0,2)$

i) line through AB

$\frac{y-1}{x-5} = \frac{1-(-1)}{5-(-2)}$ ✓ for progress

$\frac{y-1}{x-5} = \frac{2}{7}$ ✓ for correct answer.

$7(y-1) = 2(x-5)$

$7y - 7 = 2x - 10$

$2x - 7y - 3 = 0$ or $y = \frac{2}{7}x - \frac{3}{7}$

ii) line parallel through C

$2x - 7y + K = 0$ passing through $(0,2)$

ie. $2(0) - 7(2) + K = 0$ ✓ for progress

$K = 14$ ✓ for correct answer.

$\therefore$ equation of line is $2x - 7y + 14 = 0$

iii) Perp. distance $2x - 7y - 3 = 0$ to $(0,2)$

$d = \frac{|2(0) - 7(2) - 3|}{\sqrt{(2)^2 + (7)^2}}$

$= \frac{+17}{\sqrt{53}} \approx 2.33$ units. ✓ correct answer.

Q12 (b)

i) $y=\cos(x^3-2x)$

$y' = -\sin(x^3-2x)\cdot(3x^2-2)$ ✓ for use of chain rule

$=(2-3x^2)\sin(x^3-2x)$ ✓ correct differentiation

ii) $y=\dfrac{x}{1+e^x}$ $\quad u=x \quad v=1+e^x$

$u'=1 \quad v'=e^x$

$y'=\dfrac{(1+e^x)-xe^x}{(1+e^x)^2}$ or $y'=\dfrac{1+e^x(1-x)}{(1+e^x)^2}$ ✓ correct use quotient rule

✓ correct answer. in any form

i) $\int 3e^{-5x}\,dx = 3\cdot\left(-\frac{1}{5}\right)e^{-5x}+C$ ✓ Answer using e^{-5x}

$= -\frac{3}{5}e^{-5x}+C$ ✓ Correct answer.

ii) $\int x^2(1-\sqrt{x})\,dx = \int(x^2-x^{5/2})\,dx$ ✓

$=\frac{1}{3}x^3-\frac{2}{7}x^{7/2}+C$ ✓

iii) $\int_1^2 \frac{4x}{x^2+1}\,dx = \left[2\ln(x^2+1)\right]_1^2$ ✓

$=2\ln(5)-2\ln(2)$

$=2\ln\left(\frac{5}{2}\right)$ ✓

≈ 1.83258 [accept?]

Q13 (a) $y^2-4y-16x-12=0$

$y^2-4y=16x+12$

$(y-2)^2-4=16x+12$

$(y-2)^2=16x+16$

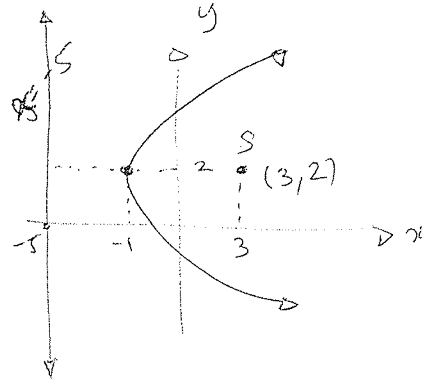
$(y-2)^2=(4)(4)(x+1)$

Vertex at $(-1,2)$

focal length $=4$.

focus point $(3,2)$

directrix $\cancel{y}$ $=-5$.



✓ finding vertex

✓ ~~correct~~ focal length

✓ graph correct. (orientation, directrix, focus point)

(b) $I=53+3\sin\left(\frac{t}{4}\right)$

(i) Period $=\dfrac{2\pi}{(1/4)}$

$=8\pi$

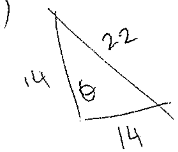
≈ 25.1 hours. ✓

(ii) Range: 53 ± 3 units ✓

or $50<I<56$

Accept: 6 units of brightness. (?)

i(c)



(i) $\cos\theta = \dfrac{14^2+14^2-22^2}{2(14)(14)}$ ✓

$\theta = 103.573\ldots^\circ$

$\theta \approx 104^\circ$ ✓

(ii) Minor segment — radians!

$A = \frac{1}{2}r^2(\theta - \sin\theta)$

$= \frac{1}{2}(14)^2\left(\frac{104\times\pi}{180} - \sin 104^\circ\right)$ ✓

$= 82.7949\ldots$ ✓

$\approx 82.8\ \text{cm}^2$

} 1 mark if done correctly but using degrees for θ

1) (i)

x	0	$\frac{\pi}{4}$	$\frac{\pi}{2}$
$\sin^3 x$	0	$\frac{1}{2\sqrt{2}}$	1

✓ using correct values.

$A \approx \dfrac{\pi/4}{2}\left[0 + 2\left(\frac{1}{2\sqrt{2}}\right) + 1\right]$ ✓ applying rule correctly

$\approx 0.67\ u^2$ (2 DP) ✓ answer

(ii) $\displaystyle\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \sin^3 x\, dx = 0$ by inspection ✓

) $1 - \frac{2}{3} + \frac{4}{9} - \frac{8}{27} + \frac{16}{81} + \ldots$

$a = 1$

$r = -\frac{2}{3}$ } ✓ for seeing values

$S_\infty = \dfrac{1}{1-\left(-\frac{2}{3}\right)}$

$= \frac{3}{5}$ ✓ for answer.

Q14.

a) $y = -x(x-5)^2 = -x^3 + 10x^2 - 25x$

i) $y' = -3x^2 + 20x - 25$

$= -(3x-5)(x-5)$

Stationary points when $y' = 0$

ie: at $x = \frac{5}{3}$ and $x = 5$ ✓

Test points:

$y'' = -6x + 20$

At $x = 5/3$,

$y'' = -6\left(\frac{5}{3}\right) + 20$

$= 10$

> 0

Hence local minimum

$y = \left(-\frac{5}{3}\right)\left(\frac{5}{3} - 5\right)^2$

$= \left(-\frac{5}{3}\right)\left(\frac{10}{3}\right)^2$

$= -\frac{500}{27} \approx 18.5$

$\left(\frac{5}{3}, -\frac{500}{27}\right)$ local min. ✓

At $x = 5$,

$y'' = -6(5) + 20$

$= -10$

< 0

Local max

$y = (-5)(5-5)^2 = 0$

$(5, 0)$ local max. ✓

Deduct 1 mark for not testing

(ii) Inflexion possible at $y'' = 0$

ie: $0 = -6x + 20$

$x = \frac{10}{3}$ ✓

$y' = -3\left(\frac{100}{9}\right) + 20\left(\frac{10}{3}\right) - 25$

$\neq 0$

wrong! Need to check both sides to show concavity changes.

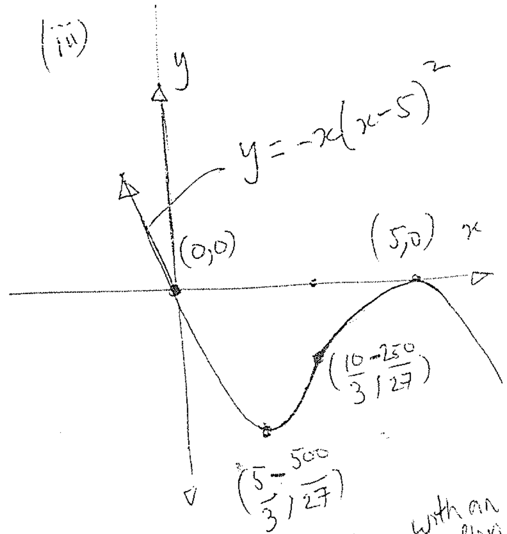
∴ Point of inflexion. ✓ check!

$y = \left(-\frac{10}{3}\right)\left(\frac{10}{3} - 5\right)^2$

$= \left(-\frac{10}{3}\right)\left(-\frac{5}{3}\right)^2$

$= -\frac{250}{27} \approx -9.26$

(iii)



✓

✓ correct shape with an inflexion

✓ stationary points consistent with (i)

... quality labelling

Q14(b)

) $V = Ae^{-kt}$

1st statement $\Rightarrow A = 5.6$ (billion) √

2nd statement $\Rightarrow$ reduce by 4.5% in 1 year:

$$0.955 = e^{-k(1)}$$

$$-\ln(0.955) = k$$ √

$$k = 0.046044$$

$$\therefore V \approx 5.6\, e^{-0.046044t}$$

t years since 2008, V in billions

i) At mid 2013, $t = 5.5$

$$V \approx 5.6\, e^{-0.046044 \times 5.5}$$

≈ 4.35 billion letters

Declines $\approx 5.6 - 4.35 \approx 1.25 \approx 1$ billion (to 1 sig. fig)

Yes – model is consistent with statement 3

√ Justify an answer using mathematical argument

14(c)

i) Test $x = 3$: $y = 2x^2 - 8x$

$= 2(3)^2 - 8(3)$

$= 6$

$y = x^3 - 7x^2 + 10$

$= (3)^3 - 7(9) + 10(3)$ √

$= 6$

$\therefore$ (3,6) satisfies both equations.

ii) Area between curves is √

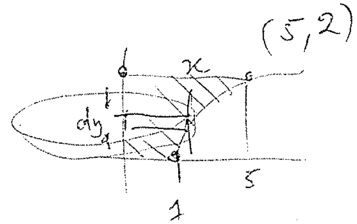
$$\int_0^3 (x^3 - 7x^2 + 10x) - (2x^2 - 8x)\,dx$$

$$= \int_0^3 (x^3 - 9x^2 + 18x)\,dx$$

$$= \left[\tfrac{1}{4}x^4 - 3x^3 + 9x^2\right]_0^3 = \tfrac{1}{4}(3)^4 - 3(27) + 9(9) = \tfrac{81}{4} \text{ units}^2$$ √

Q15.

i) $y = \sqrt{x-1}$ $\quad V = \pi \int_{y=0}^{y=2} x^2\,dy$



$y^2 = x - 1$

$x = y^2 + 1$ ——— √

$\therefore x^2 = (y^2+1)^2$

$= y^4 + 2y^2 + 1$

$$V = \pi \int_0^2 (y^4 + 2y^2 + 1)\,dy$$ — √

$$= \pi\left[\tfrac{1}{5}y^5 + \tfrac{2}{3}y^3 + y\right]_0^2$$ — √

$$= \pi\left[\tfrac{32}{5} + \tfrac{16}{3} + 2\right]$$

$$= \tfrac{206}{15}\pi \text{ units}^3$$ — √

ii) In $\triangle ABN$, $\triangle ACM$, ← students not writing this line

$AB = AC$ (given)

$\angle BAN = \angle CAM$ (common angle)

$\angle ABN = \angle ACM$ (given)

$\therefore \triangle ABN \equiv \triangle ACM$ (two angles equal, corresponding sides equal) AAS

√2 full proof or √1 makes progress

Now $BM = BA - MA$

and $MA = NA$ (corresponding sides in cong. triangles $\triangle ABN$, $\triangle ACM$)

$\therefore BM = BA - NA$

and $BA = AC$ (given)

$\therefore BM = AC - NA$ ($AC = AN + NC$)

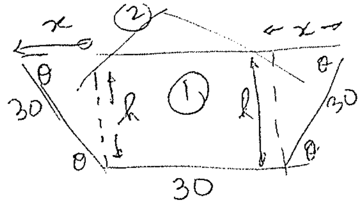
$= CN$

as required.

√2 full consistent proof or √1 makes progress

Q15. (c)

(i)



$x = 30\cos\theta$
$y = 30\sin\theta$ } ✓

Area (1) $= b \times h = 30 \times 30\sin\theta$

Area (2) $= 2 \times \frac{1}{2} \times b \times h$ — ✓

$= 2 \times \frac{1}{2} \times 30\sin\theta \, 30\cos\theta$

Total area $= 900\sin\theta + 900\sin\theta\cos\theta$ — ✓

$= 900(\sin\theta + \sin\theta\cos\theta)$ as required.

(ii) $\theta = 900(\sin\theta + \sin\theta\cos\theta)$ or product rule

$\frac{1}{2}\sin 2\theta$
$= \cos 2\theta$
$= \cos^2\theta - \sin^2\theta$

$\theta' = 900(\cos\theta + \cos\theta \cdot \cos\theta - \sin\theta \cdot \sin\theta)$ ✓

$= 900(\cos\theta + \cos^2\theta - \sin^2\theta)$

$= 900(\cos\theta + 2\cos^2\theta - 1)$ ✓

$= 900(2\cos\theta - 1)(\cos\theta + 1)$

Stationary points at ✓ find two possible solutions

$2\cos\theta - 1 = 0$ or $\cos\theta = -1$

$\cos\theta = \frac{1}{2}$ $\quad \theta = -180°$

$\theta = \frac{\pi}{3}$ $\quad$ reject this solution - not a valid fold for this problem

Check point:

$\theta'' = 900(-\sin\theta + 2\cos\theta \cdot (-\sin\theta))$

At $\theta = \frac{\pi}{3}$, $\theta'' = 900\left(-\frac{\sqrt{3}}{2} + 2\left(\frac{1}{2}\right)\left(-\frac{\sqrt{3}}{2}\right)\right) < 0$ Concave down

∴ Maximum cross sectional area when $\theta = \frac{\pi}{3}$ ✓ check solution

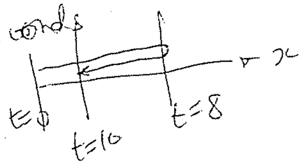
Q16.

i) $\dot{x} = 0.1(t^3 - 27t^2 + 132t + 160)$ $\quad 0 \le t \le 10$

$x = 0.1\int(t^3 - 27t^2 + 132t + 160)\,dt$ ✓

$= 0.1\left(\frac{1}{4}t^4 - 9t^3 + 66t^2 + 160t\right) + C$

Cheetah runs to right $t=0 \to t=8$ seconds
then turns back



At $t=0$, $x = C$

At $t=8$, $x = 0.1\left[\frac{1}{4}(8)^4 - 9(8)^3 + 66(64) + 160(8)\right] + C$

$= 192 + C$

At $t=10$, $x = 0.1\left(\frac{1}{4}(1000) - 9(1000) + 66(100) + 160(10)\right) + C$

$= 170 + C$

Distance travelled $= 192 + (192 - 170)$ ✓ for progress

$= 214$m ✓ for answer.

(ii) $\ddot{x} = 0.1(3t^2 - 54t + 132)$ ✓

When $\ddot{x} = 0$,

$3t^2 - 54t + 132 = 0$

$t = \frac{54 \pm \sqrt{54^2 - 4(3)(132)}}{6}$

≈ 2.9 or 15.1 seconds

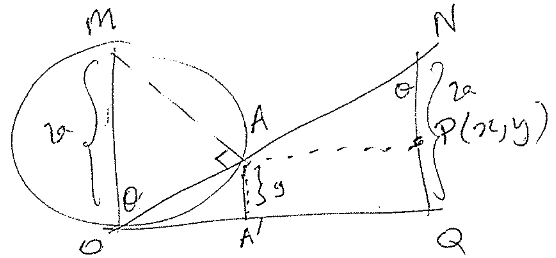
15.1 is not in the range.

2.9 seconds is a maximum (by inspection of $\dot{x}$ graph)

∴ Max velocity cheetah is 2.9 seconds (to 1 DP).

✓ Answer clearly explained must reject 15.1 second ow

Q16(b)



15

(i) $\angle ONQ = \theta$ (alt. angles in || lines)

In $\triangle ONQ$,

$\tan\theta = \dfrac{OQ}{NQ}$

$\tan\theta = \dfrac{x}{2a}$

$\therefore x = 2a\tan\theta$ ✓

(ii) In $\triangle MOA$,

$\cos\theta = \dfrac{OA}{OM}$

$\cos\theta = \dfrac{OA}{2a}$ (A) ✓

Height y is AA',

$\angle OAA' = \theta$ (alt angles in || lines)

In $\triangle OAA'$, $\cos\theta = \dfrac{y}{OA}$ ✓

$y = \cos\theta \cdot OA$

$= 2a\cos^2\theta$ (using (A))

(iii) $x^2 = 4a^2\tan^2\theta$ (using (i))

$x^2 + 4a^2 = 4a^2\tan^2\theta + 4a^2$

$= 4a^2(1+\tan^2\theta)$

$x^2 + 4a^2 = 4a^2\sec^2\theta$

$x^2 + 4a^2 = \dfrac{4a^2}{\cos^2\theta}$

$\therefore \cos^2\theta = \dfrac{4a^2}{x^2+4a^2}$

$\therefore \dfrac{y}{2a} = \dfrac{4a^2}{x^2+4a^2}$ (using (ii)

$y = \dfrac{8a^3}{x^2+4a^2}$ as required

✓ mark progress

✓ mark answer.

(16)

Q16: (c)

(i) $f(x) = Ax^3 + Bx^2 + Cx + D$

$\int_{-k}^{k} (Ax^3 + Bx^2 + Cx + D)\,dx = \left[\frac{A}{4}x^4 + \frac{B}{3}x^3 + \frac{C}{2}x^2 + Dx\right]_{-k}^{k}$

$= \frac{2B}{3}k^3 + 2Dk$ (even degrees cancel out)

$= 2k\left(\frac{B}{3}k^2 + D\right)$ (as required) /1

(ii) $p = -k + \frac{1}{3}(2k)$

$= -\frac{k}{3}$

$q = k - \frac{1}{3}(2k)$

$= \frac{k}{3}$ /1

$f(-k) + 3f\left(-\frac{k}{3}\right) + 3f\left(\frac{k}{3}\right) + f(k)$

$= -Ak^3 + Bk^2 - Ck + D + 3\left(\frac{-Ak^3}{27} + \frac{B}{9}k^2 - \frac{C}{3}k + D\right)$

$+ 3\left(\frac{Ak^3}{27} + \frac{B}{9}k^2 + \frac{C}{3}k + D\right) + Ak^3 + Bk^2 + Ck + D$

$= 2Bk^2 + 2D + \frac{2B}{3}k^2 + 6D$ /1

$= \frac{8Bk^2}{3} + 8D$

$= 8\left(\frac{B}{3}k^2 + D\right)$

(iii) We can make:

$\cancel{\left(\frac{k}{4}\right)\cdot\left(f(-k) + 3f\left(-\frac{k}{3}\right) + 3f\left(\frac{k}{3}\right) + f(k)\right) = \left(\frac{k}{4}\right)(8)\left(\frac{B}{3}k^2 + D\right)}$

$\cancel{= 2k\left(\frac{B}{3}k^2 + D\right)}$

$= \int_{-k}^{k} f(x)\,dx$ (i)

18

(iii) From (ii)

$$f(-k) + 3f\left(-\frac{k}{3}\right) + 3f\left(\frac{k}{3}\right) + f(k) = 8\left(\frac{B}{3}k^2 + D\right)$$

Multiply by $\frac{k}{4}$

$$\frac{k}{4}\left(f(-k) + 3f\left(-\frac{k}{3}\right) + 3f\left(\frac{k}{3}\right) + f(k)\right) = 2k\left(\frac{B}{3}k^2 + D\right)$$

$$= \int_{-k}^{k} f(x)\,dx \quad \text{(from (i))}$$

ie: We can write

$$\int_{-k}^{k} f(x)\,dx = \frac{k}{4}\left(f(-k) + 3f\left(-\frac{k}{3}\right) + 3f\left(\frac{k}{3}\right) + f(k)\right)$$

Shifting $-k \le x \le k$ to $a \le x \le b$

points are now a, $\frac{2a+b}{3}$, $\frac{a+2b}{3}$, b

and $k = \left(\frac{b-a}{2}\right)$

$$\therefore \int_a^b f(x)\,dx = \left(\frac{b-a}{2}\right)\left(\frac{1}{4}\right)\left(f(a) + 3f\left(\frac{2a+b}{3}\right) + 3f\left(\frac{a+2b}{3}\right) + f(b)\right)$$

$$= \frac{b-a}{8}\left[f(a) + 3f\left(\frac{2a+b}{3}\right) + 3f\left(\frac{a+2b}{3}\right) + f(b)\right]$$

✓1 makes progress

✓1 correct expression using a & b only.