PENRITH HIGH SCHOOL

MATHEMATICS EXTENSION 2
2013

HSC Trial

Assessor: Mr Ferguson

General Instructions:

- Reading time – 5 minutes
- Working time – **3 hours**
- Write using black or blue pen. Black pen is preferred
- Board-approved calculators may be used.
- A table of standard integrals is provided at the back of this paper.
- Show all necessary working in Questions 11 – 16.
- Work on this question paper will not be marked.

Total marks – 100

SECTION 1 – Pages 2 - 7

10 marks

- Attempt Questions 1 – 10
- Allow about 15minutes for this section.

SECTION 2 – Pages 8 - 13

90 marks

- Attempt Questions 11 – 16
- Allow about 2 hours 45 minutes for this section.

Section1

Question	Mark
1-3	
4-5	
6-7	
8	
9	
10	
Total	/10

Section 2

Question	Mark
11	/15
12	/15
13vol	/7
13graph	/8
14conic	/12
14circle	/3

Question	Mark
15motion	/5
15poly	/7
15 harder ext	/3
16complex	/6
16 harder ext	/9

Total	/100
%	

This paper MUST NOT be removed from the examination room

Student Name: ………………………………

Section I

10 marks
Attempt Questions 1–10
Allow about 15 minutes for this section

Use the multiple-choice answer sheet for Questions 1–10.

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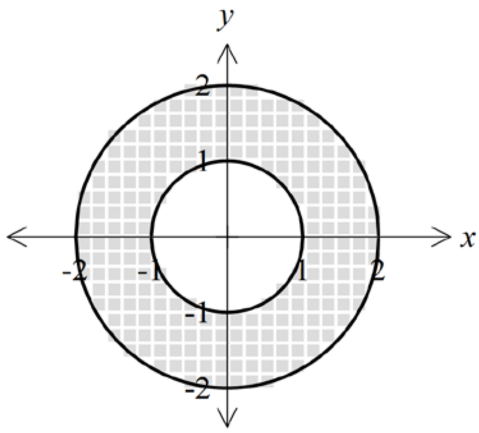
1 Consider the Argand diagram below.



The inequality that represents the shaded area is:

(A) $0 \le |z| \le 2$

(B) $1 \le |z| \le 2$

(C) $0 \le |z-1| \le 2$

(D) $1 \le |z-1| \le 2$

2 Let $z = 1 + 2i$ and $w = -2 + i$. The value of $\dfrac{5}{iw}$ is:

(A) $-1-2i$

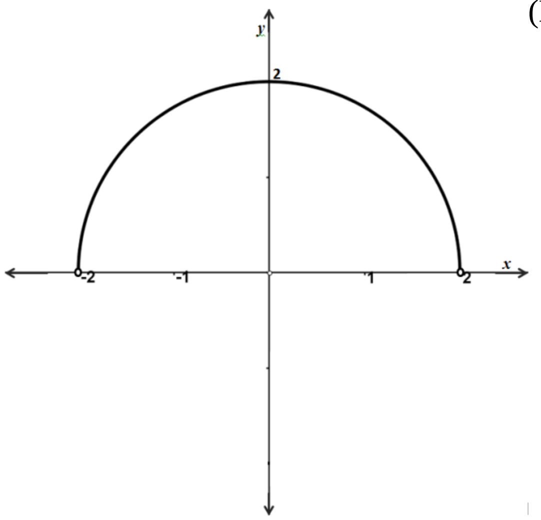
(B) $-1+2i$

(C) $1-2i$

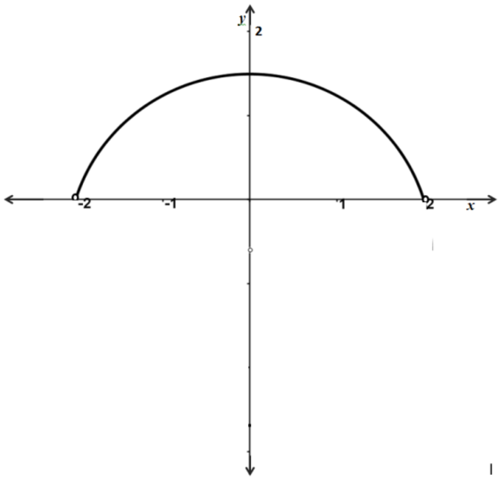
(D) $1+2i$

3 The locus of z if $\arg(z-2)-\arg(z+2)=\dfrac{\pi}{4}$ is best shown as:

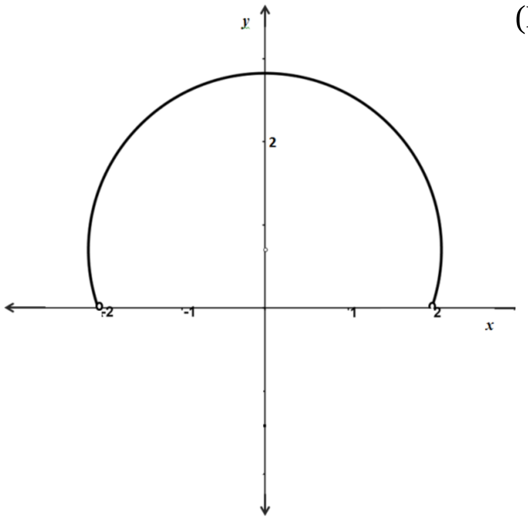
(A)



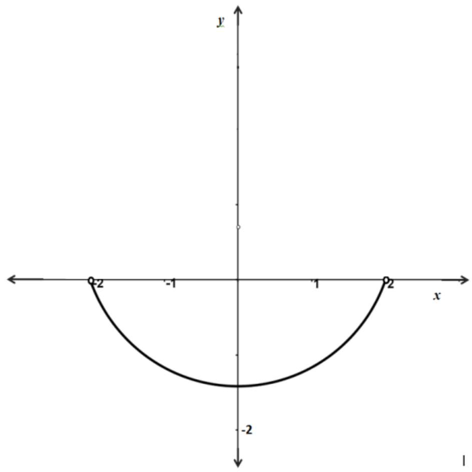
(B)



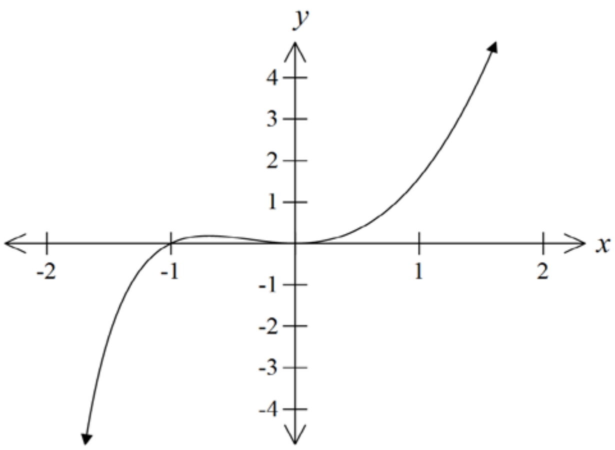
(C)



(D)

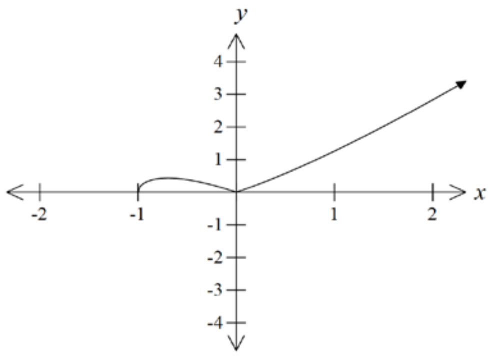


4 The diagram shows the graph of the function $y = f(x)$.

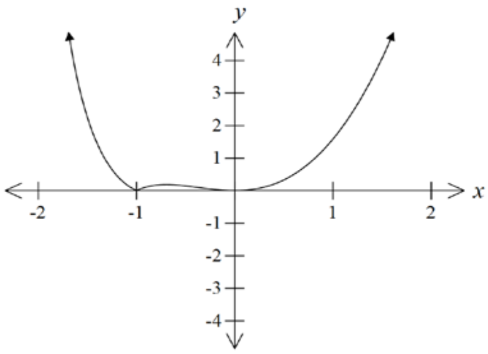


The diagram that shows the graph of the function $y = f(x)^2$ is:

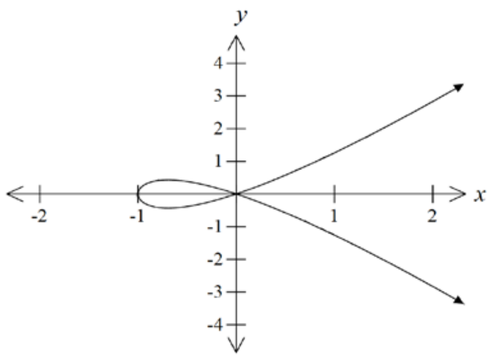
(A)



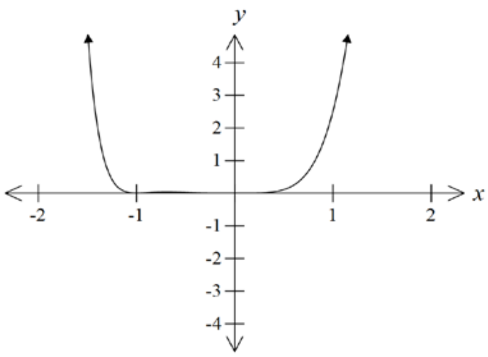
(B)



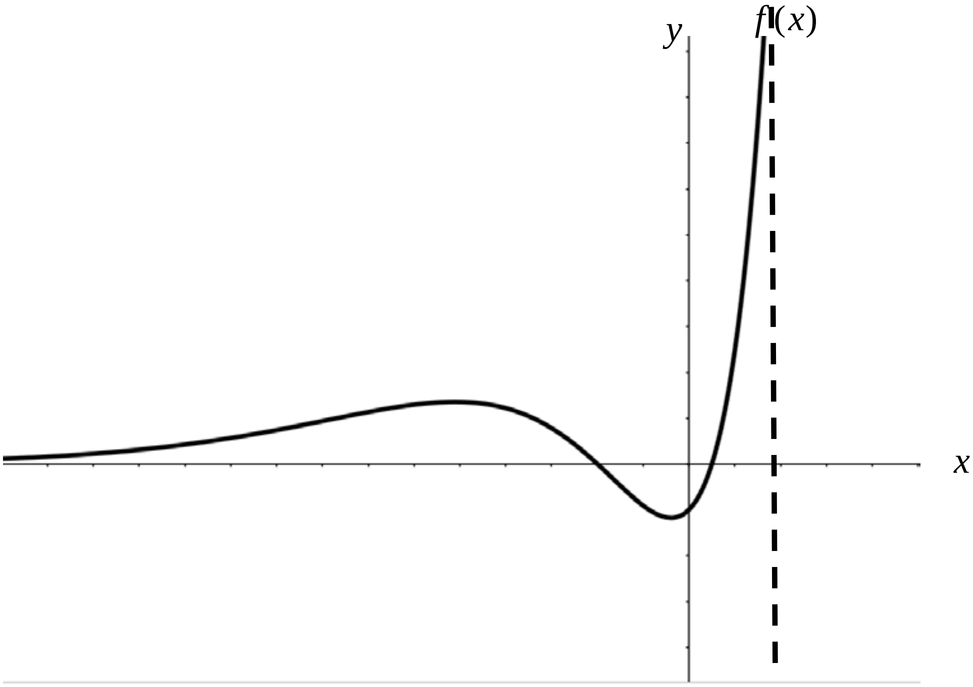
(C)



(D)



5 The function $y = f(x)$ is drawn below.



Which of the following is an incorrect statement?

(A) $y = f(x)$ has two asymptotes only

(B) $y = f(x)$ is continuous everywhere in its domain

(C) $y = f(x)$ has exactly one point of inflexion

(D) $y = f(x)$ is differentiable everywhere in its domain.

6 The values of the real numbers p and q that makes $1-i$ a root of the equation $z^3 + pz + q = 0$ are:

(A) $p = -2$ and $q = -4$

(B) $p = -2$ and $q = 4$

(C) $p = 2$ and $q = -4$

(D) $p = 2$ and $q = 4$

7 Let $P(x)$ be a polynomial of degree $n > 0$ such that $P(x) = (x-\alpha)^r Q(x)$, where $r \geq 2$ and α is a real number. $Q(x)$ is a polynomial with real coefficients of degree $q > 0$. Which of the following is the incorrect statement?

(A) $n \leq r + q$

(B) $P(x)$ changes sign around the root $x = \alpha$

(C) Let N_r be the number of real roots of $P(x)$ and N_c the number of complex roots of $P(x)$. Then $r \leq N_r \leq n$ and $0 \leq N_c \leq q$.

(D) Roots of $P(x)$ are conjugate one another.

8 The eccentricity of the ellipse with the equation $\frac{x^2}{4} + \frac{y^2}{3} = 1$ is:

(A) $\frac{1}{4}$

(B) $\frac{1}{2}$

(C) $\frac{3}{4}$

(D) $\frac{9}{16}$

9 A particle of mass m falls from rest under gravity and the resistance to its motion is mkv, where v is its speed and k is a positive constant. Which of the following is the correct expression for the velocity?

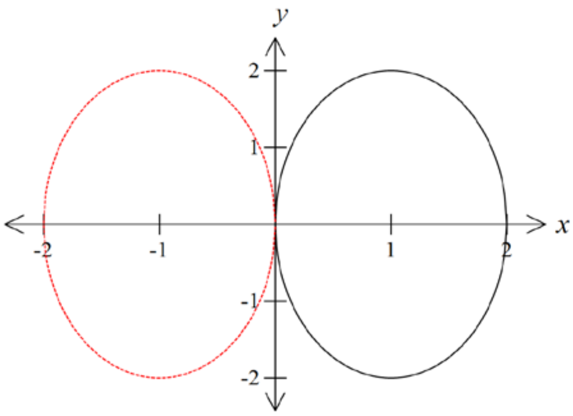
(A) $v = \frac{g}{k}\left(1 - e^{-kt}\right)$

(B) $v = \frac{g}{k}\left(1 + e^{-kt}\right)$

(C) $v = \frac{g}{k}\left(1 - e^{kt}\right)$

(D) $v = \frac{g}{k}\left(1 + e^{kt}\right)$

10 The region enclosed by the ellipse $(x-1)^2+\frac{y^2}{4}=1$ is rotated about the y axis to form a solid.



What is the correct expression for volume of this solid using the method of slicing?

(A) $V=\int_{-2}^{2}\pi\sqrt{1-y^2}\,dy$

(B) $V=\int_{-2}^{2}2\pi\sqrt{1-y^2}\,dy$

(C) $V=\int_{-2}^{2}\pi\sqrt{4-y^2}\,dy$

(D) $V=\int_{-2}^{2}2\pi\sqrt{4-y^2}\,dy$

Section II

90 marks
Attempt Questions 11–16
Allow about 2 hours and 45 minutes for this section

Answer each question in a SEPARATE writing booklet. Extra writing booklets are available.

In Questions 11–16, your responses should include relevant mathematical reasoning and/or calculations.

Question 11 (15 marks) Use a SEPARATE writing booklet.

(a) The complex number z is given by $z = -1 + \sqrt{3}i$

(i) Show that $z^2 = 2\overline{z}$ **2**

(ii) Evaluate $|z|$ and $Arg(z)$ **2**

(b) Calculate the product of the roots of the following equation in the form $a + ib$ **2**

$(3+2i)z^2 - (1-2i)z + (6-i) = 0$

(c) Find the complex square roots of $7 - 6\sqrt{2}i$ giving your answers in the form $x + iy$, where x and y are real. **3**

(d) (i) Express $z = \sqrt{3} + i$ in modulus/argument form. **3**

(ii) Show that $z^7 + 64z = 0$

(e) The points A, B, C, D on the Argand diagram represent the complex numbers a, b, c, d respectively. If $a + c = b + d$ and $a - c = i(b - d)$ find what type of quadrilateral $ABCD$ is. **3**

Question 12 (15 marks) Use a SEPARATE writing booklet.

(a) Find $\int_0^3 \frac{\sqrt{x}}{1+x}\,dx$ (Hint: let $u^2 = x$) **3**

(b) By using a suitable trigonometric substitution show that $\int \frac{dx}{\sqrt{1-x^2}} = \sin^{-1} x + c$ **2**

where c is some constant

(c) (i) Find real numbers A, B and C such that $\int \frac{x+6}{(x+1)(x^2+9)}\,dx \equiv \int \frac{A}{x+1} + \int \frac{Bx+C}{x^2+9}$ **2**

(ii) Hence, find $\int \frac{x+6}{(x+1)(x^2+9)}\,dx$ **2**

(d) Using the substitution $t = \tan\frac{x}{2}$, evaluate $\int_0^{\frac{\pi}{2}} \frac{dx}{5+3\cos x}$ **3**

(e) For $n \geq 0$ let, **3**

$$I_n = \int_0^{\frac{\pi}{2}} x^n \sin x\ dx$$

Show that for $n \geq 2$, $I_n = n\left(\frac{\pi}{2}\right)^{n-1} - n(n-1)I_{n-2}$

Question 13 (15 marks) Use a SEPARATE writing booklet.

(a) The area bounded by the curve $y = 12x - x^2$, the x axis, $x = 2$ and $x = 10$ is rotated about the y axis to form a solid. By using the method of cylindrical shells calculate the volume of the solid. **4**

(b) The base of a solid is the segment of the parabola $x^2 = 4y$ cut off by the line $y = 2$. Each cross section perpendicular to the y axis is a right angled isosceles triangle with the hypotenuse in the base of the solid. Find the volume of the solid. **3**

(c) Consider the function $f(x) = \dfrac{e^x - 1}{e^x + 1}$

(i) Show that $f(x)$ is an odd function **1**

(ii) Show that the function is always increasing **2**

(iii) Find $f'(0)$ **1**

(iv) Discuss the behaviour of $f(x)$ as $x \to \pm\infty$ **2**

(v) Sketch the graph of $y = f(x)$ **2**

Question 14 (15 marks) Use a SEPARATE writing booklet.

(a) For the ellipse $\frac{x^2}{4}+\frac{y^2}{3}=1$

(i) Find the eccentricity **1**

(ii) Find the coordinates of the foci S and S' **1**

(iii) Find the coordinates of the directices. **1**

(iv) Sketch the curve showing foci and directices. **1**

P is an arbitrary point on this ellipse

(v) Prove that the sum of the distances SP and $S'P$ is independent of P **2**

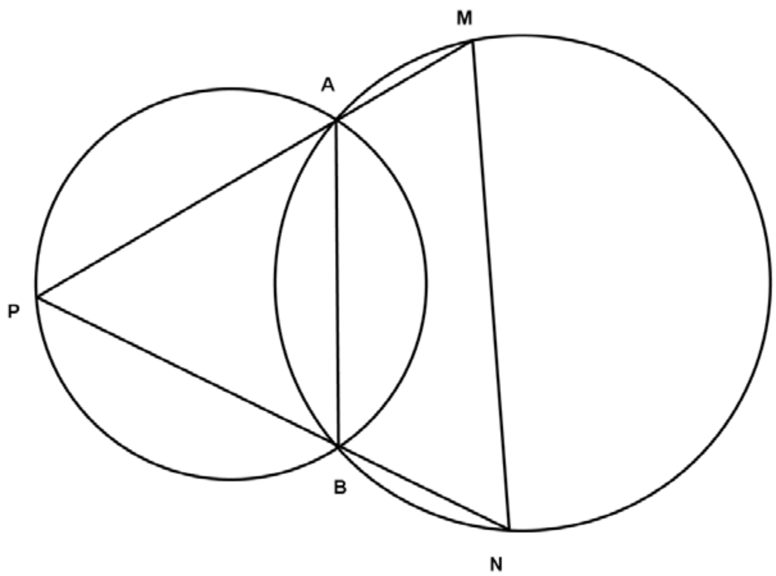
(b) (i) Find the slope of the tangent to the ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$ at the point **2**

$P(a\cos\theta, b\sin\theta)$

(ii) Hence show that the equation of this tangent is $\frac{x\cos\theta}{a}+\frac{y\sin\theta}{b}=1$. **2**

(iii) If the point $P(a\cos\theta, b\sin\theta)$ is on the ellipse in quadrant one, find the minimum area of the triangle made by this tangent and the coordinate axes. **2**

(c) Two fixed circles intersect at AB. P is a variable point on one circle.



Copy or trace the diagram into your writing booklet.

(i) Let $\angle APB=\theta$, explain why θ is a constant. **1**

(ii) Prove that MN is of a constant length. **2**

Question 15 (15 marks) Use a SEPARATE writing booklet.

(a) A body of unit mass falls under gravity through a resisted medium. The body falls from rest. The resistance to its motion is $\frac{1}{100}v^2$ Newtons where v metres per second is the speed of the body when it has fallen a distance x metres.

(i) Show that the equation of motion of the body is $\ddot{x} = g - \frac{1}{100}v^2$, where g is the magnitude of the acceleration due to gravity. (Note: Draw a diagram!) **1**

(ii) Show that the terminal speed, V_T , is given by $V_T = 10\sqrt{g}$ **1**

(iii) Show that $V^2 = V_T{}^2\left(1 - e^{\frac{-x}{50}}\right)$ **3**

(b) $x^3 - 6x^2 + 9x + k = 0$ has two equal roots.

(i) Show that $k = -4$ is a possible value for k **2**

(ii) Solve $x^3 - 6x^2 + 9x - 4 = 0$ **2**

(c) $\alpha, \beta,$ and γ are the roots of $x^3 + px^2 + qx + r = 0$

Find

(i) $\alpha^2 + \beta^2 + \gamma^2$ **1**

(ii) $\alpha^3 + \beta^3 + \gamma^3$ **2**

(d)

(i) Prove that $x^2 + x + 1 \geq 0$ for all real x **1**

(ii) Hence or otherwise, prove that $a^2 + ab + b^2 \geq 0$ **2**

Question 16 (15 marks) Use a SEPARATE writing booklet.

(a) Let $w = \cos\frac{2\pi}{7} + i\sin\frac{2\pi}{7}$

(i) Show that w^k is a solution of $z^7 - 1 = 0$, where k is an integer. **2**

(ii) Prove that $w + w^2 + w^3 + w^4 + w^5 + w^6 = -1$ **2**

(iii) Hence show that $\cos\frac{2\pi}{7} + \cos\frac{4\pi}{7} + \cos\frac{6\pi}{7} = -\frac{1}{2}$ **2**

(b) The Bernoulli polynomials $B_n(x)$ are defined by $B_0(x) = 1$ and for n=1, 2, 3, …,

$$\frac{dB_n(x)}{dx} = nB_{n-1}(x), \text{ and}$$

$$\int_0^1 B_n(x)dx = 0$$

Thus

$$B_1(x) = x - \frac{1}{2}$$

$$B_2(x) = x^2 - x + \frac{1}{6}$$

$$B_3(x) = x^3 - \frac{3}{2}x^2 + \frac{1}{2}x$$

(i) Show that $B_4(x) = x^2(x-1)^2 - \frac{1}{30}$ **2**

(ii) Show that, for $n \geq 2, B_n(1) - B_n(0) = 0$ **1**

(iii) Show by mathematical induction, that for $n \geq 1$ **3**

$$B_n(x+1) - B_n(x) = n\,x^{n-1},$$

(iv) Hence show that for $n \geq 1$and any positive integer k **2**

$$n\sum_{m=0}^{k} m^{n-1} = B_n(k+1) - B_n(0)$$

(v) Hence deduce that $\sum_{m=0}^{135} m^4 = 9134962308$ **1**

STANDARD INTEGRALS

$$\int x^n\,dx = \frac{1}{n+1}x^{n+1}, \quad n \neq -1;\ x \neq 0, \quad \text{if } n < 0$$

$$\int \frac{1}{x}\,dx = \ln x, \quad x > 0$$

$$\int e^{ax}\,dx = \frac{1}{a}e^{ax}, \quad a \neq 0$$

$$\int \cos ax\,dx = \frac{1}{a}\sin ax, \quad a \neq 0$$

$$\int \sin ax\,dx = -\frac{1}{a}\cos ax, \quad a \neq 0$$

$$\int \sec^2 ax\,dx = \frac{1}{a}\tan ax, \quad a \neq 0$$

$$\int \sec ax \tan ax\,dx = \frac{1}{a}\sec ax, \quad a \neq 0$$

$$\int \frac{1}{a^2+x^2}\,dx = \frac{1}{a}\tan^{-1}\frac{x}{a}, \quad a \neq 0$$

$$\int \frac{1}{\sqrt{a^2-x^2}}\,dx = \sin^{-1}\frac{x}{a}, \quad a > 0, \quad -a < x < a$$

$$\int \frac{1}{\sqrt{x^2-a^2}}\,dx = \ln\left(x + \sqrt{x^2-a^2}\right), \quad x > a > 0$$

$$\int \frac{1}{\sqrt{x^2+a^2}}\,dx = \ln\left(x + \sqrt{x^2+a^2}\right)$$

Note: $\ln x = \log_e x, \quad x > 0$

SECTION I

MULTIPLE CHOICE

(1) B

(2) $\frac{5}{i(-2+i)} = \frac{5}{-2i+i^2}$

$= \frac{5}{-1-2i} \times \frac{-1+2i}{-1+2i}$

$= \frac{-5+10i}{5}$

$= -1+2i$

B

(3) C

(4) D

(5) C

(6) Roots $(1-i), (1+i), \gamma$

$\Sigma\alpha = -\frac{b}{a} = 0$

$\Sigma\alpha\beta = \frac{c}{a} = p$

$\Sigma\alpha\beta\gamma = -\frac{d}{a} = -q$

$\Sigma\alpha = (1-i) + (1+i) + \gamma = 0$

$\Rightarrow 2 + \gamma = 0$

$\Rightarrow \gamma = -2$

$\Sigma\alpha\beta\gamma = (1-i)(1+i).(-2)$

$2.-2 = -q$

$\therefore q = 4$

B

$\Sigma\alpha\beta \quad (1-i)(1+i) + -2(1-i) + -2(1+i)$

$2 - 2 + 2i + -2 - 2i$

$-2 = p \qquad \therefore p = -2 \quad q = 4$

(7) D

(8) $b^2 = a^2(1-e^2)$

$3 = 4(1-e^2)$

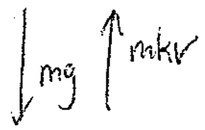
$\frac{3}{4} = 1 - e^2$

$e^2 = 1 - \frac{3}{4}$

$e = \frac{1}{2}$

B

(9)



$\ddot{x} = g - kv$

$\frac{dv}{dt} = g - kv$

$\frac{dt}{dv} = -\frac{1}{k} \cdot \frac{-k}{g-kv}$

① $- \quad t = -\frac{1}{k}\ln(g-kv) + c$

② $- \quad 0 = -\frac{1}{k}\ln(g) + c \qquad t=0 \; v=0$

①-② $\quad t = -\frac{1}{k}\ln\left(\frac{g-kv}{g}\right)$

$-kt = \ln\left(\frac{g-kv}{g}\right)$

$e^{-kt} = \frac{g-kv}{g}$

$ge^{-kt} = g - kv$

$kv = g - ge^{-kt}$

$v = \frac{g}{k}(1 - e^{-kt})$

(A)

(10) $\pi \int x_2{}^2 - x_1{}^2 \, dy$

$V = \pi \int_{-2}^{2} (x_2{}^2 - x_1{}^2) \, dy$

$(x-1)^2 + \frac{y^2}{4} = 1$

$x^2 - 2x + 1 + \frac{y^2}{4} = 1$

$x^2 - 2x + \frac{y^2}{4} = 0$

$\Sigma\alpha = -2 \qquad x_1 + x_2 = -2$

$\Sigma\alpha\beta = \frac{y^2}{4} \qquad x_1 x_2 = \frac{y^2}{4}$

$x_2{}^2 - x_1{}^2 = (x_2 - x_1)(x_2 + x_1)$

$x_2 - x_1 = \sqrt{(x_2 - x_1)^2}$

$= \sqrt{(x_2 + x_1)^2 - 4x_1x_2}$

$= \sqrt{(-2)^2 - 4\left(\frac{y^2}{4}\right)}$

$x_2 - x_1 = \sqrt{4 - y^2}$

$\therefore x_2{}^2 - x_1{}^2 = \sqrt{4-y^2} \times 2$

$= 2\sqrt{4-y^2}$

$\int_{-2}^{2} 2\pi \sqrt{4-y^2} \, dy$

(D)

SECTION II

(11)

a)(i) $z = -1 + \sqrt{3}i$

$z^2 = (-1 + \sqrt{3}i)^2$

$= 1 - 2\sqrt{3}i - 3$

$= -2 - 2\sqrt{3}i$

$\bar{z} = -1 - \sqrt{3}i$

$2\bar{z} = -2 - 2\sqrt{3}i$

$= z^2$

(ii) $|z| = \sqrt{(-1)^2 + (\sqrt{3})^2}$

$= \sqrt{1+3}$

$= 2$

$\text{Arg } z = 120$ or $\frac{2\pi}{3}$

b) $\alpha\beta = \frac{c}{a}$

$= \frac{6-i}{3+2i} \times \frac{3-2i}{3-2i}$

$= \frac{18 - 12i - 3i - 2}{9+4}$

$= \frac{16 - 15i}{13}$

$= \frac{16}{13} - \frac{15}{13}i$

c) $z^2 = 7 - 6\sqrt{2}i$

$x^2 - y^2 + 2ixy = 7 - 6\sqrt{2}i$

$x^2 - y^2 = 7$

$2xy = -6\sqrt{2}$ *

$(x^2+y^2)^2 = (x^2-y^2)^2 + 4x^2y^2$

$= 49 + 72$

$(x^2+y^2)^2 = 121$

$x^2 + y^2 = 11$ — ①

$x^2 - y^2 = 7$ — ②

①+② $2x^2 = 18$

①-② $2y^2 = 4$

$x^2 = 9 \quad \therefore x = \pm 3$

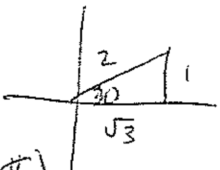
$y^2 = 2 \quad \therefore y = \pm 2$

check in *

$2xy = -6\sqrt{2}$

$\therefore \pm(3 - 2i)$

d) i) $z = \sqrt{3} + i$



$z = 2\left(\cos\frac{\pi}{6} + i\sin\frac{\pi}{6}\right)$

(ii) $z^7 = 2^7\left(\cos\frac{7\pi}{6} + i\sin\frac{7\pi}{6}\right)$

$= 128\left(\cos -\frac{5\pi}{6} + i\sin -\frac{5\pi}{6}\right)$

$= 128\left(-\cos\frac{\pi}{6} - i\sin\frac{\pi}{6}\right)$

$= -128\left(\cos\frac{\pi}{6} + i\sin\frac{\pi}{6}\right)$

$= -64.2\left(\cos\frac{\pi}{6} + i\sin\frac{\pi}{6}\right)$

$= -64z$

(e) $a + c = b + d$

$\frac{a+c}{2} = \frac{b+d}{2}$

$\therefore$ midpoint AC = midpoint BD

$\therefore$ diagonals AC, BD bisect eachother

$a - c = i(b - d)$

$\therefore$ diagonal AC and BD are equal and perpendicular

$\therefore$ ABCD is a square

(12) a) $\int_0^3 \frac{\sqrt{x}}{1+x}\,dx$

Let $u^2 = x$

$2u\frac{du}{dx} = 1$

$\frac{du}{dx} = \frac{1}{2u}$ $\quad 2u\,du = dx$

$dx =$

$= \int \frac{u}{1+u^2} \cdot 2u\,du$

$= \int \frac{2u^2}{1+u^2}\,du$

$= 2\int 1 - \frac{1}{1+u^2}$

$= 2\left[u - \tan^{-1}(u)\right]_0^{\sqrt{3}}$

$= 2\left[\sqrt{3} - \frac{\pi}{3}\right]$

$= \frac{6\sqrt{3} - 2\pi}{3}$

b) $\int \frac{dx}{\sqrt{1-x^2}}$

Let $x = \sin\theta$

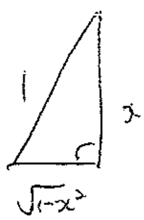
$\frac{dx}{d\theta} = \cos\theta$

$dx = \cos\theta\,d\theta$

$= \int \frac{1}{\sqrt{1-\sin^2\theta}}\cos\theta\,d\theta$

$= \int \frac{\cos\theta}{\cos\theta}\,d\theta$

$= \int 1\,d\theta$



$= \theta + C$

$= \sin^{-1}x + C$

c (i) $\int \frac{x+6}{(x+1)(x^2+9)}\,dx$

$(x^2+9)A + (x+1)(Bx+C) = x+6$

Let $x = -1$

$10A = 5$

$A = \frac{1}{2}$

Let $x = 0$

$9\left(\frac{1}{2}\right) + 1(C) = 6$

$9 + 2C = 12$

$2C = 3$

$C = \frac{3}{2}$

Let $x = 1$

$\frac{10}{2} + 2\left(B + \frac{3}{2}\right) = 7$

$2\left(B + \frac{3}{2}\right) = 2$

$B + \frac{3}{2} = 1$

$B = -\frac{1}{2}$

$A = \frac{1}{2} \quad B = -\frac{1}{2} \quad C = \frac{3}{2}$

(ii) $\int \frac{x+6}{(x+1)(x^2+9)}\,dx$

$\frac{1}{2}\int \frac{1}{x+1} + \int \frac{-\frac{1}{2}x + \frac{3}{2}}{x^2+9}$

$= \frac{1}{2}\ln(x+1) + -\frac{1}{4}\int \frac{2x-6}{x^2+9}$

$= \frac{1}{2}\ln(x+1) + -\frac{1}{4}\left[\frac{2x}{x^2+9} - 6\int \frac{1}{x^2+9}\right]$

$= \frac{1}{2}\ln(x+1) - \frac{1}{4}\ln(x^2+9) + \frac{3}{2} \times \frac{1}{3}\tan^{-1}\left(\frac{x}{3}\right)$

$= \frac{1}{2}\ln(x+1) - \frac{1}{4}\ln(x^2+9) + \frac{1}{2}\tan^{-1}\left(\frac{x}{3}\right) + C$

(d) Let $t = \tan\frac{x}{2}$

$\therefore dt = \frac{1}{2}\sec^2\frac{x}{2}\,dx$

$= \frac{1+t^2}{2}\,dx$ $\qquad x=0\ t=0$

$\qquad x=\frac{\pi}{2}\ t=1$

$dx = \frac{2\,dt}{1+t^2}$

$$\int \frac{dx}{5+3\cos x} = \int_0^1 \frac{\frac{2\,dt}{1+t^2}}{5+3\left(\frac{1-t^2}{1+t^2}\right)}$$

$$= \int_0^1 \frac{2\,dt}{5+5t^2+3-3t^2}$$

$$= \int_0^1 \frac{2\,dt}{8+2t^2}$$

$$= \int_0^1 \frac{1}{4+t^2}$$

$$= \frac{1}{2}\left[\tan^{-1}\frac{t}{2}\right]_0^1$$

$$= \frac{1}{2}\left[\tan^{-1}\frac{1}{2}\right]$$

(e)

$$I_n = \int_0^{\frac{\pi}{2}} x^n \sin x\,dx$$

$u = x^n \qquad u' = nx^{n-1}$

$v' = \sin x \qquad v = -\cos x$

$$= \left[-x^n\cos x\right]_0^{\frac{\pi}{2}} + n\int_0^{\frac{\pi}{2}} x^{n-1}\cos x\,dx$$

$$= n\int_0^{\frac{\pi}{2}} x^{n-1}\cos x\,dx$$

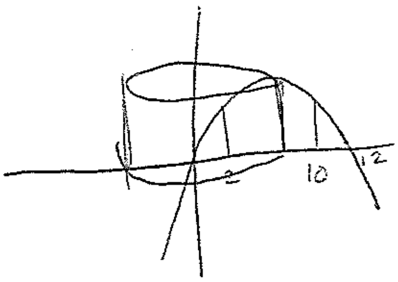
$$\int_0^{\frac{\pi}{2}} x^{n-1}\cos x\,dx = \left[x^{n-1}\sin x\right]_0^{\frac{\pi}{2}} - (n-1)\int_0^{\frac{\pi}{2}} x^{n-2}\sin x\,dx$$

$$= \left(\frac{\pi}{2}\right)^{n-1} - (n-1)I_{n-2}$$

$$\therefore I_n = n\left(\frac{\pi}{2}\right)^{n-1} - n(n-1)I_{n-2}$$

(13)

a)



$$V = \int 2\pi x y \, dx$$

$$= 2\pi \int_{-2}^{10} x(12x - x^2)\, dx$$

$$= 2\pi \int_{2}^{10} 12x^2 - x^3 \, dx$$

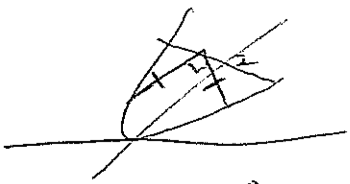
$$= 2\pi \left[4x^3 - \frac{x^4}{4} \right]_2^{10}$$

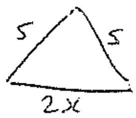
$$= 2\pi \left[\left(4000 - \frac{10000}{4}\right) - \left(32 - 4\right) \right]$$

$$= 2\pi [1500 - 28]$$

$$= 2944\pi \text{ units}^2$$

b)





$$2s^2 = 4x^2$$

$$s^2 = 2x^2$$

$$s = x\sqrt{2}$$

area of triangle is $\frac{1}{2} b \times h$

$$\frac{1}{2} s^2$$

$$= x^2$$

$$= 4y$$

$$V = \int_0^2 4y \, dy$$

$$\left[2y^2\right]_0^2 = 8 \text{ cubic units}$$

(c)(i) $f(x) = \dfrac{e^x - 1}{e^x + 1}$

$$f(-x) = \frac{e^{-x} - 1}{e^{-x} + 1} \times \frac{e^x}{e^x}$$

$$= \frac{1 - e^x}{1 + e^x}$$

$$= \frac{-e^x - 1}{e^x + 1}$$

$$= -f(x)$$

(ii) $f'(x) = \dfrac{e^x(e^x+1) - (e^x - 1)e^x}{(e^x+1)^2}$

$$= \frac{2e^x}{(e^x+1)^2}$$

but $e^x > 0$

$(e^x + 1)^2 > 1$

$\therefore f'(x) > 0$

$\therefore f(x)$ is alway increasing

(iii) $f'(0) = \dfrac{2e^0}{(e^0+1)^2}$

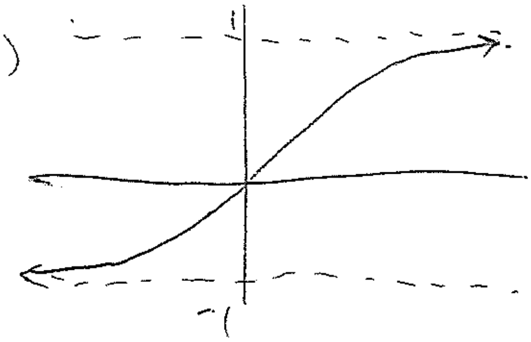
$$= \frac{2}{2^2} = \frac{1}{2}$$

(iv) $f(x) = \dfrac{e^x - 1}{e^x + 1}$

$$= \frac{1 - e^{-x}}{1 + e^{-x}} \rightarrow 1^- \text{ as } x \rightarrow \infty$$

$$f(x) = \frac{e^x - 1}{e^x + 1} \rightarrow -1^+ \text{ as } x \rightarrow -$$

(v)



(14) $\frac{x^2}{4}+\frac{y^2}{3}=1 \quad a=2 \quad b=\sqrt{3}$

a)(i) $b^2=a^2(1-e^2)$

$3=4(1-e^2)$

$\frac{3}{4}=1-e^2$

$e^2=\frac{1}{4}$

$e=\frac{1}{2}$

(ii) $S=(ae,0)$

$=(1,0)$

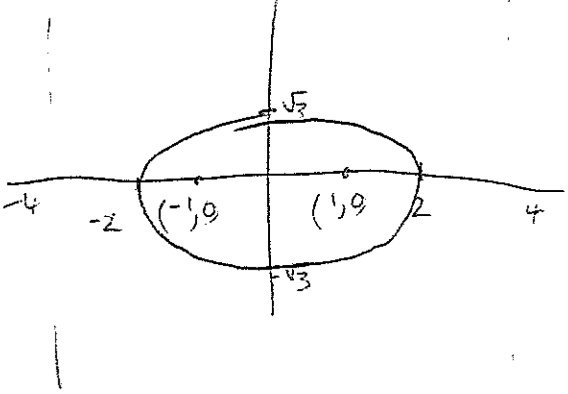
$S'=(-ae,0)$

$=(-1,0)$

(iii) $x=\pm\frac{a}{e}$

$\Rightarrow x=\pm 4$

(iv)



(v) $PS+PS'=e\,PD+e\,PD'$

since $e=\frac{PS}{PD}=\frac{PS'}{PD'}$

$=e(PD+PD')$

$=8e$

$\therefore$ independent of P

(b) $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$

$b^2x^2+a^2y^2=a^2b^2$

$2b^2x+2a^2y\frac{dy}{dx}=0$

$\frac{dy}{dx}=\frac{-b^2x}{a^2y}$

at $P(a\cos\theta, b\sin\theta)$

$\frac{dy}{dx}=\frac{-ab^2\cos\theta}{a^2b\sin\theta}$

$=\frac{-b\cos\theta}{a\sin\theta}$

(ii) $y-b\sin\theta=-\frac{b}{a}\frac{\cos\theta}{\sin\theta}(x-a\cos\theta)$

$\frac{x\cos\theta}{a}+\frac{y\sin\theta}{b}=1$

(iii) when $x=0$ $\quad y=\frac{b}{\sin\theta}$

when $y=0$ $\quad x=\frac{a}{\cos\theta}$

so $A\left(0, \frac{b}{\sin\theta}\right)$

$B\left(\frac{a}{\cos\theta}, 0\right)$

Area of ΔAOB

$= \frac{1}{2}(OB \cdot OA)$

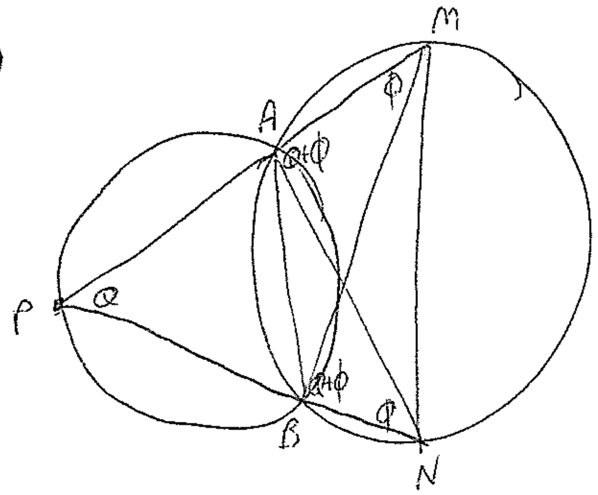
$= \frac{ab}{2\sin\theta\cos\theta}$

$= \frac{ab}{\sin 2\theta}$

min when $\sin 2\theta$ max

$\sin 2\theta = 1$ ie when $\theta = \frac{\pi}{4}$

(c)



(i) since AB is a fixed chord

$\therefore \theta$ is a constant since angle subtended from same chord AB are equal.

(ii) $\angle\phi$ is fixed (angle subtended by same chord AB)

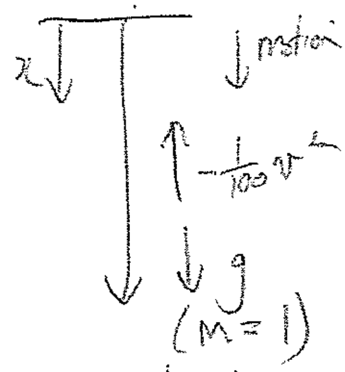
$\angle MAN = \angle MBN = \theta + \phi$ (both are exterior angles to ΔPAN and ΔPBM respectively)

$=$ constant.

$\therefore$ MN is a constant since fixed circumference angles subtends fixed chords in the same circle.

(15)

(a)



$F = mg - \frac{1}{100}v^2$

$\ddot{x} = g - \frac{1}{100m}v^2$

$m = 1$

$\ddot{x} = g - \frac{1}{100}v^2$

(ii) when terminal

$\Sigma F = 0$

$\therefore g - \frac{1}{100}V_T^2 = 0$

$V_T^2 = 100g$

$= \sqrt{100g}$

$= 10\sqrt{g}$

(iii) $v\frac{dv}{dx} = g - \frac{1}{100}v^2$

$\int \frac{v\,dv}{g - \frac{1}{100}v^2} = \int dx$

$L.H.S = \int \frac{100v}{100g - v^2}$

$= \int \frac{100v}{V_T^2 - v^2}$

$= -\frac{100}{2}\ln(V_T^2 - v^2)$

$-50\ln(V_T^2 - v^2) = x + c$

$V_T^2 - v^2 = Ae^{-\frac{x}{50}}$

$v^2 = V_T^2 - Ae^{-\frac{x}{50}}$

when $x = 0$ $v = 0$

$0 = V_T^2 - A$

$A = V_T^2$

$V^2 = V_T^2(1 - e^{-\frac{x}{50}})$

b) $P'(x) = 3x^2 - 12x + 9$

$3(x^2 - 4x + 3) = 0$

$3(x-3)(x-1) = 0$

$x = 1$ or $x = 3$

(i) $P(1) = 0$

$1 - 6 + 9 + k = 0$

$k = -4$

(ii) $x^3 - 6x^2 + 9x - 4 = 0$

$(x-1)^2 = x^2 - 2x + 1$

$$\begin{array}{r} x - 4 \\ x^2 - 2x + 1 \overline{)\,x^3 - 6x^2 + 9x - 4} \\ x^3 - 2x^2 \quad\quad\quad \\ \hline -4x^2 + 9x - 4 \\ -4x^2 + 9x - 4 \end{array}$$

$x = 4$

$\therefore x = 1, 1, 4.$

c(i) $\alpha^2+\beta^2+\gamma^2 = (\alpha+\beta+\gamma)^2 - 2(\alpha\beta+\alpha\gamma+\beta\gamma)$

$= -p^2 - 2q$

$= p^2 - 2q$ ✓.

(ii) $x^3 = -px^2 - qx - r$

$\alpha^3 = -p\alpha^2 - q\alpha - r$

$\beta^3 = -p\beta^2 - q\beta - r$

$\gamma^3 = -p\gamma^2 - q\gamma - r$

$\alpha^3+\beta^3+\gamma^3 = -p(\alpha^2+\beta^2+\gamma^2) - q(\alpha+\beta+\gamma) - 3r$

$= -p(p^2-2q) - q(-p) - 3r$

$= -p^3 + 3pq - 3r$.

d) $x^2+x+1 \geq 0$.

$\Delta = b^2 - 4ac$

$1 - 4 = <0$. ∴ always true.

(ii) $a^2+ab+b^2 = b^2\left(\frac{a^2}{b^2} + \frac{ab}{b^2} + \frac{b^2}{b^2}\right)$

$= b^2\left(\left(\frac{a}{b}\right)^2 + \left(\frac{a}{b}\right) + 1\right)$

$= b^2(x^2+x+1)$ if $x = \frac{a}{b}$

≥ 0.

16 a (i)

$w = \cos\frac{2\pi}{7} + i\sin\frac{2\pi}{7}$

$w^k = \cos\frac{2\pi k}{7} + i\sin\frac{2k\pi}{7}$

$\therefore z^7 - 1 = (w^k)^7 - 1$

$= (\cos 2k\pi + i\sin 2k\pi) - 1$

$= (1+0) - 1 = 0$

(ii) $w + w^2 + w^3 + \dots w^6$

G.P $= \frac{a(1-r^n)}{1-r}$

$= \frac{w(1-w^6)}{1-w}$

$= \frac{w - w^7}{1-w}$

$= \frac{w-1}{1-w}$

$= -1$

(iii) $w + w^6 = 2\cos\frac{2\pi}{7}$

$w^2 + w^5 = 2\cos\frac{4\pi}{7}$

$w^3 + w^4 = 2\cos\frac{6\pi}{7}$

$2\cos\frac{2\pi}{7} + 2\cos\frac{4\pi}{7} + 2\cos\frac{6\pi}{7} = -1$

$\cos\frac{2\pi}{7} + \cos\frac{4\pi}{7} + \cos\frac{6\pi}{7} = -\frac{1}{2}$

(b)

(i) $B_4(x) = 4\int B_3(x)\,dx$

$= \int 4x^3 - 6x^2 + 2x\,dx$

$= x^4 - 2x^3 + x^2 + c$

Now $\int_0^1 x^4 - 2x^3 + x^2 + c\,dx = 0$

$\therefore \left[\frac{1}{5}x^5 - \frac{1}{2}x^4 + \frac{1}{3}x^3 + cx\right]_0^1 = 0$

$\therefore c = \frac{1}{2} - \frac{1}{3} - \frac{1}{5}$

$= -\frac{1}{30}$

$\therefore B_4(x) = x^4 - 2x^3 + x^2 - \frac{1}{30}$

$= x^2(x-1)^2 - \frac{1}{30}$

(ii) $B_n(1) - B_n(0)$

$= \int_0^1 B_{n-1}(x)\,dx$

$= 0$ by definition

ie $B_n(1) - B_n(0) = 0$

If $n=1$: $\int_0^1 B_0(x)\,dx = \int_0^1 dx$

$= 0$

(iii) Let $S(n)$ be the statement that $B_n(x+1) - B_n(x) = nx^{n-1}$ for some positive integer n.

Now $B_1(x+1) - B_1(x)$

$= x + 1 - \frac{1}{2} - (x - \frac{1}{2})$

$= 1$

$= 1 . x^{1-1}$

Hence $S(1)$ true

Let k be some positive integer for which $S(k)$ true

ie $B_k(x+1) - B_k(x) = kx^{k-1}$

Consider

$\frac{d}{dx}\left[B_{k+1}(x+1) - B_{k+1}(x)\right]$

$= (k+1)B_k(x+1) - (k+1)B_k(x)$

$= (k+1)\left[B_k(x+1) - B_k(x)\right]$

$= (k+1)k.x^{k-1}$ by assumption

$\therefore B_{k+1}(x+1) - B_{k+1}(x) = (k+1) x^k$

(iv) $B_n(1) - B_n(0) = n\, 0^{n-1}$

$B_n(2) - B_n(1) = n\, 1^{n-1}$

$B_n(3) - B_n(2) = n\, 2^{n-1}$

$\vdots \qquad \vdots \qquad \vdots$

$B_n(k) - B_n(k-1) = n(k-1)^{n-1}$

$B_n(k+1) - B_n(k) = n k^{n-1}$

Sum of L.H.S -

$B_n(k+1) - B_n(0)$

Sum of R.H.S

$n(0^{n-1} + 1^{n-1} + \dots k^{n-1})$

$= n \sum_{m=0}^{k} m^{n-1}$

(v) $\sum_{m=0}^{135} m^4 = \frac{1}{5}(B_5(136) - B_5(0))$

$B_5(x) = 5 \int B_4(x)\, dx$

$B_5(x) = x^5 - \frac{5}{2}x^4 + \frac{5}{3}x^3 - \frac{x}{6} + C$

$= \frac{1}{5}(136^5 - \frac{5}{2} \times 136^4 + \frac{5}{3} \times 136^3 - \frac{1}{6} \times 136 + C - C)$

$= 9134962308$