



# THE KING'S SCHOOL

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## 2008 Higher School Certificate Trial Examination

### Mathematics Extension 2

#### General Instructions

- Reading time – 5 minutes
- Working time – 3 hours
- Write using black or blue pen
- Board-approved calculators may be used
- A table of standard integrals is provided at the back of this paper
- All necessary working should be shown in every question
- Answer each question in a separate booklet

#### Total marks – 120

- Attempt Questions 1-8
- All questions are of equal value

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**Total marks – 120**

**Attempt Questions 1-8**

**All questions are of equal value**

Answer each question in a SEPARATE writing booklet. Extra writing booklets are available.

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**Question 1 (15 marks)** Use a SEPARATE writing booklet.

**Marks**

(a) Find  $\int \tan^2 x \, dx$  **2**

(b) Find  $\int \frac{x}{x+1} \, dx$  **2**

(c) (i) Let  $F(x)$  be a primitive function of  $f(x)$ . Hence, or otherwise, show that  $\int_0^a f(x) \, dx = \int_0^a f(a-x) \, dx$  **1**

(ii) Show that  $\int_0^\pi x \sin x \, dx = \frac{\pi}{2} \int_0^\pi \sin x \, dx$  **2**

(iii) Use integration by parts to evaluate  $\int_0^\pi x^2 \cos x \, dx$  **3**

(d) (i) Find  $\int \frac{dt}{(2t+1)^2 + 1}$  **1**

(ii) Use the substitution  $t = \tan \frac{\theta}{2}$  to show that

$$\int_0^{\frac{\pi}{2}} \frac{d\theta}{2\sin\theta - \cos\theta + 3} = \tan^{-1}\left(\frac{1}{2}\right)$$
 **4**

**End of Question 1**

(a) (i) Find  $|\sqrt{7} + \sqrt{33}i|$  1

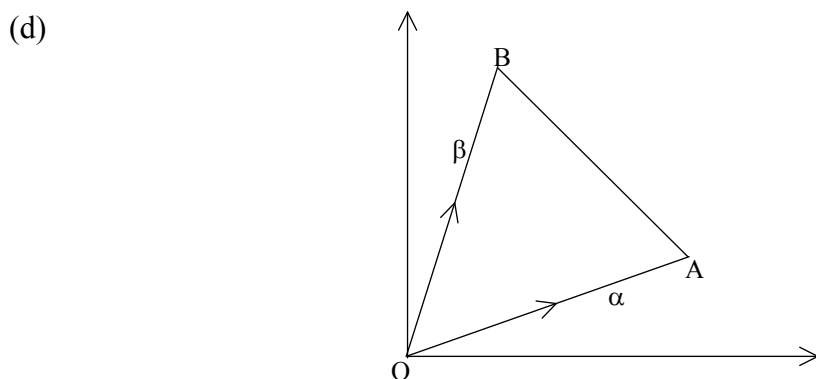
(ii)  $x + iy = \frac{\sqrt{7} + \sqrt{33}i}{3 - i}$   
Find the value of  $x^2 + y^2$  2

(b) Precisely show on the Argand diagram the locus of the complex numbers  $z$  such that  $|z - i| = 1$  and  $|z| \leq 1$  hold simultaneously. 3

(c) Let  $z = 1 - \cos 2\theta + i \sin 2\theta$ ,  $0 < \theta < \frac{\pi}{2}$

(i) Show that  $z = 2\sin\theta (\sin\theta + i\cos\theta)$  2

(ii) Hence find  $|z|$  and  $\arg z$  2



The diagram shows the equilateral triangle OAB in the complex plane.

O is the origin and points A, B represent the complex numbers  $\alpha$ ,  $\beta$ , respectively.

Let  $v = \cos \frac{\pi}{3} + i \sin \frac{\pi}{3}$

(i) Write down the complex number  $\overrightarrow{BA}$  1

(ii) Show that  $\alpha = v(\alpha - \beta)$  2

(iii) Prove that  $\alpha^2 + \beta^2 = \alpha\beta$  2

**End of Question 2**

(a) For the hyperbola  $(y + 1)^2 - x^2 = 1$ , prove that  $\frac{d^2y}{dx^2} = \frac{1}{(y + 1)^3}$  **4**

(b) Let  $P(x) = x^4 - 2Ax^3 + B$ , where  $A \neq 0$

$P(x) = 0$  has the roots  $\alpha, \beta, \gamma$  and  $\alpha + \beta + \gamma$

(i) Deduce that  $B = A^4$  **3**

(ii) Find, in simplest form,  $\alpha^2 + \beta^2 + \gamma^2$  **3**

(c) Let  $(1 + x)^{2008} = u_1 + u_2 + \dots + u_k + u_{k+1} + \dots + u_{2009}$ ,  $x > 0$

(i) Show that  $\frac{u_{k+1}}{u_k} = \frac{2009 - k}{k} \cdot x$  **2**

(ii) The middle term in the expansion of  $(1 + x)^{2008}$  is the greatest term.

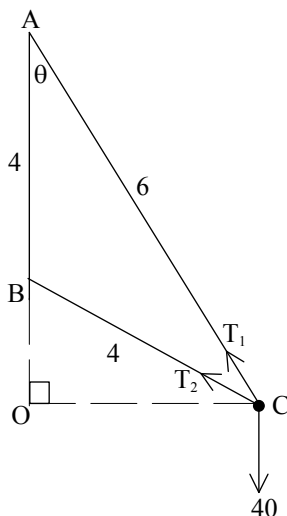
Deduce that  $\frac{1004}{1005} < x < \frac{1005}{1004}$  **3**

**End of Question 3**

- 
- (a) (i) Sketch the hyperbola  $\frac{x^2}{4} - \frac{y^2}{5} = 1$  showing its foci, directrices and asymptotes. 4
- (ii) A particular solid has as its base the region bounded by the hyperbola  $\frac{x^2}{4} - \frac{y^2}{5} = 1$  and the line  $x = 4$ .
- Cross-sections perpendicular to this base and the  $x$  axis are equilateral triangles.
- Find the volume of this solid. 4
- (b) A particle moves on the  $x$  axis according to the acceleration equation of motion  $\ddot{x} = x$ . Initially the particle is at the origin with velocity  $v = 2$ .
- (i) Explain why the velocity will always increase. 1
- (ii) By integration, prove that  $v = \sqrt{x^2 + 4}$  2
- (iii) By using the table of standard integrals, or otherwise, find the displacement  $x$  as a function of time  $t$ . 4

**End of Question 4**

(a)



Two pieces of light inextensible string AC of length 6 metres and BC of length 4 metres are attached at two points A and B, respectively. B is 4 metres vertically below A.

At C a mass of 4 kg is attached to the strings and this mass rotates in uniform circular motion of 3 rad/s about a point O which is vertically below B. Take  $10 \text{ m/s}^2$  as the acceleration due to gravity.

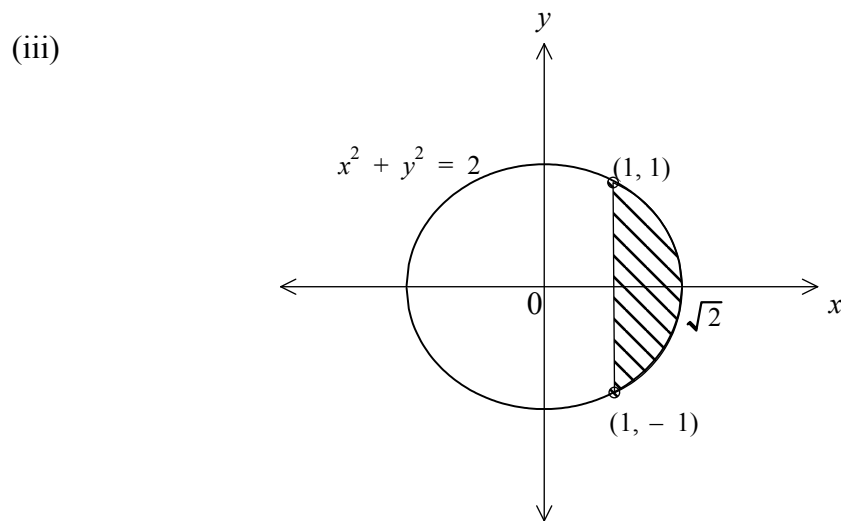
Let the tensions in the strings AC and BC be  $T_1$  Newtons and  $T_2$  Newtons, respectively, and let  $\angle BAC = \theta$

- |   |          |
|---|----------|
| (i) Show that $\cos \theta = \frac{3}{4}$   | <b>1</b> |
| (ii) By resolving forces at C in the vertical direction, show that $6T_1 + T_2 = 320$ | <b>3</b> |
| (iii) Find the tensions in the strings.   | <b>3</b> |

**Question 5 continues on the next page**

(b) (i) Show that  $\int_1^{\sqrt{2}} x\sqrt{2-x^2} \, dx = \frac{1}{3}$  2

(ii) By considering the circle  $x^2 + y^2 = 2$ , or otherwise, show that  $\int_1^{\sqrt{2}} \sqrt{2-x^2} \, dx = \frac{\pi}{4} - \frac{1}{2}$  2



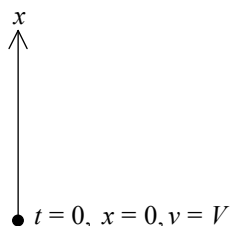
The minor segment of the circle  $x^2 + y^2 = 2$  bounded by the chord  $x = 1$  is revolved about that chord.

Use the method of cylindrical shells to find the volume of the solid generated. 4

**End of Question 5**



(a)



A particle of mass  $m$  is projected vertically upwards with speed  $V$  in a medium where there is a resistance  $mgk^2v^2$  when  $v$  is its speed.  $g$  is the acceleration due to gravity and  $k$  is a positive constant.

Take  $x = 0$  and  $v = V$  when  $t = 0$

The particle reaches a maximum height  $X$  when the time is  $T$ .

(i) Show that the equation of motion is given by  $\ddot{x} = -g(1 + k^2v^2)$  1

(ii) Show that  $X = \frac{1}{2gk^2} \ln(1 + k^2V^2)$  4

(iii) Show that  $T = \frac{1}{gk} \tan^{-1}(kV)$  3

(iv) If the only force acting on the particle is due to gravity the equations of motion are:

$$\ddot{x} = -g$$

$$\dot{x} = -gt + V$$

$$x = -\frac{gt^2}{2} + Vt$$

**[DO NOT SHOW THESE]**

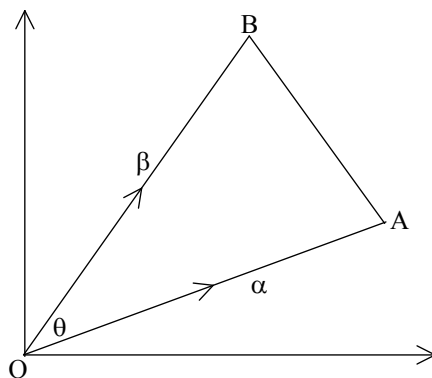
Deduce that  $\lim_{k \rightarrow 0} \frac{\ln(1 + k^2V^2)}{k^2} = V^2$  2

**Question 6 continues next page**

- (b) (i) Use the results  $z + \bar{z} = 2\operatorname{Re}(z)$  and  $|z|^2 = z\bar{z}$  for complex numbers  $z$  to show that  $|\alpha|^2 + |\beta|^2 - |\alpha - \beta|^2 = 2\operatorname{Re}(\alpha\bar{\beta})$

3

(ii)



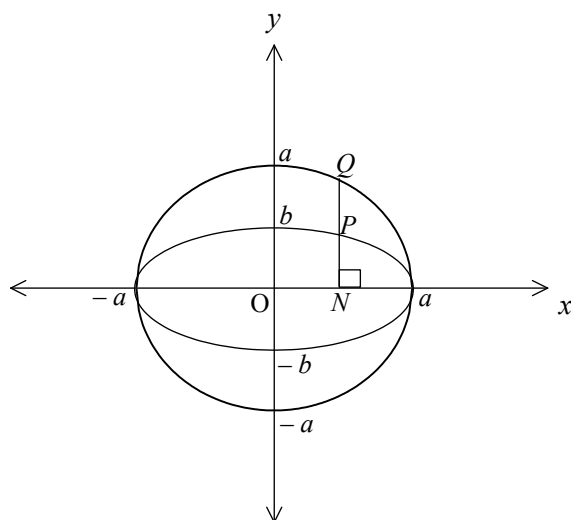
The diagram shows the angle  $\theta$  between the complex numbers  $\alpha$  and  $\beta$ .

Prove that  $|\alpha| |\beta| \cos\theta = \operatorname{Re}(\alpha\bar{\beta})$

2

**End of Question 6**

(a)



The diagram shows the circle  $x^2 + y^2 = a^2$  and the ellipse  $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ ,  $a > b > 0$

$P(a \cos \theta, b \sin \theta)$ ,  $\theta \neq -\frac{\pi}{2}, \frac{\pi}{2}$ , is a point on the ellipse.  $PN$  is perpendicular to the  $x$  axis at  $N$  and meets the circle at  $Q$  in the same quadrant.

$O$  is the origin.

- |   |          |
|---|----------|
| (i) Write down the coordinates of $Q$ .   | <b>1</b> |
| (ii) Show that the equation of the tangent at $P(a \cos \theta, b \sin \theta)$ on the ellipse is $\frac{\cos \theta}{a}x + \frac{\sin \theta}{b}y = 1$ | <b>2</b> |
| (iii) Hence, or otherwise, find the equation of the tangent at the point $Q$ on the circle.   | <b>1</b> |
| (iv) The tangents at $P$ and $Q$ meet at $T$ . Prove that $ON \cdot OT = a^2$ .   | <b>2</b> |

**Question 7 continues on the next page**

(b) Let  $u_n = \int_0^{\frac{\pi}{2}} \cos^n \theta \, d\theta$ ,  $n = 0, 1, 2, \dots$

(i) Explain why  $u_n < u_{n-1} < u_{n-2}$  1

(ii) Prove that  $u_n = \frac{n-1}{n} u_{n-2}$ ,  $n = 2, 3, 4, \dots$  3

(iii) Deduce that  $\lim_{n \rightarrow \infty} u_n = \lim_{n \rightarrow \infty} u_{n-1}$  1

(iv) Use (ii) to show that  $n u_n u_{n-1} = \frac{\pi}{2}$ ,  $n = 1, 2, 3, \dots$  2

(v) Given that  $\int_0^{\frac{\pi}{2}} \cos^{11} \theta \, d\theta = \frac{256}{693}$ ,  
evaluate  $\int_0^{\frac{\pi}{2}} \cos^{10} \theta \, d\theta$  1

(vi) Find an approximate value of  $\int_0^{\frac{\pi}{2}} \cos^{2008} \theta \, d\theta$  1

**End of Question 7**

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(a) Let  $\frac{(x+a)(x+b)(x+c)}{(x-a)(x-b)(x-c)} \equiv k + \frac{p}{x-a} + \frac{q}{x-b} + \frac{t}{x-c}$

(i) Explain why  $k = 1$  **1**

(ii) Show that  $p = \frac{2a(a+b)(a+c)}{(a-b)(a-c)}$  and write down expressions for  $q$  and  $t$ . **3**

(iii) Hence, or otherwise, prove that

$$\frac{(a+b)(a+c)}{(a-b)(a-c)} + \frac{2b(b+c)}{(b-a)(b-c)} + \frac{2c(c+b)}{(c-a)(c-b)} = 1$$
 **2**

**Question 8 continues on the next page**

(b) The roots of  $x^4 + x^3 + 2x^2 + 3x + 1 = 0$  are  $\alpha, \beta, \gamma, \delta$

(i) Find a polynomial with the roots  $\frac{1}{\alpha}, \frac{1}{\beta}, \frac{1}{\gamma}, \frac{1}{\delta}$ . 2

(ii) Hence, or otherwise, show that

$$\left(\alpha + \frac{1}{\alpha}\right) + \left(\beta + \frac{1}{\beta}\right) + \left(\gamma + \frac{1}{\gamma}\right) + \left(\delta + \frac{1}{\delta}\right) = -4 \quad 1$$

(iii) Explain why the equation  $x^4 + 4x^3 + Ax^2 + Bx + C = 0$  for some  $A, B, C$  has the roots  $\alpha + \frac{1}{\alpha}, \beta + \frac{1}{\beta}, \gamma + \frac{1}{\gamma}, \delta + \frac{1}{\delta}$  1

(iv) Hence state the eight roots of the equation

$$\left(x + \frac{1}{x}\right)^4 + 4\left(x + \frac{1}{x}\right)^3 + A\left(x + \frac{1}{x}\right)^2 + B\left(x + \frac{1}{x}\right) + C = 0 \quad 1$$

(v) Use the equation in (iii) to state the four roots of the equation

$$Cx^4 + Bx^3 + Ax^2 + 4x + 1 = 0 \quad 1$$

(vi) By multiplying both sides by  $x^4$ , the equation in (iv) could be expressed as  $(x^2 + 1)^4 + 4x(x^2 + 1)^3 + Ax^2(x^2 + 1)^2 + Bx^3(x^2 + 1) + Cx^4 = 0$

[DO NOT SHOW THIS]

Hence, by using the polynomial found in (i) and another suitable equation, prove that  $B = 0$ . 2

(vii) Evaluate  $\left(\alpha + \frac{1}{\alpha}\right)^{-1} + \left(\beta + \frac{1}{\beta}\right)^{-1} + \left(\gamma + \frac{1}{\gamma}\right)^{-1} + \left(\delta + \frac{1}{\delta}\right)^{-1}$  1

**End of Examination Paper**

## Standard Integrals

$$\int x^n dx = \frac{1}{n+1} x^{n+1}, \quad n \neq -1; \quad x \neq 0, \text{ if } n < 0$$

$$\int \frac{1}{x} dx = \ln x, \quad x > 0$$

$$\int e^{ax} dx = \frac{1}{a} e^{ax}, \quad a \neq 0$$

$$\int \cos ax dx = \frac{1}{a} \sin ax, \quad a \neq 0$$

$$\int \sin ax dx = -\frac{1}{a} \cos ax, \quad a \neq 0$$

$$\int \sec^2 ax dx = \frac{1}{a} \tan ax, \quad a \neq 0$$

$$\int \sec ax \tan ax dx = \frac{1}{a} \sec ax, \quad a \neq 0$$

$$\int \frac{1}{a^2 + x^2} dx = \frac{1}{a} \tan^{-1} \frac{x}{a}, \quad a \neq 0$$

$$\int \frac{1}{\sqrt{a^2 - x^2}} dx = \sin^{-1} \frac{x}{a}, \quad a > 0, \quad -a < x < a$$

$$\int \frac{1}{\sqrt{x^2 - a^2}} dx = \ln \left( x + \sqrt{x^2 - a^2} \right), \quad x > a > 0$$

$$\int \frac{1}{\sqrt{x^2 + a^2}} dx = \ln \left( x + \sqrt{x^2 + a^2} \right)$$

Note:  $\ln x = \log_e x, \quad x > 0$



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## 2008 Higher School Certificate Trial Examination

### Mathematics Extension 2

Question	(Marks)	Complex Numbers	Functions	Integration	Conics	Mechanics
1	(15)			15		
2	(15)	15				
3	(15)		15			
4	(15)			(a)(ii) 4	(a)(i) 4	(b) 7
5	(15)			(b) 8		(a) 7
6	(15)	(b) 5				(a) 10
7	(15)			(b) 9	(a) 6	
8	(15)		15			
Total	(120)	20	30	36	10	24



Question 1

$$(a) \int \tan^2 x \, dx = \int \sec^2 x - 1 \, dx = \tan x - x$$

$$(b) \int \frac{x}{x+1} \, dx = \int \frac{x+1-1}{x+1} \, dx = \int 1 - \frac{1}{1+x} \, dx = x - \ln(1+x)$$

$$(c) \quad (i) \int_0^a f(a-x) \, dx = \left[ -F(a-x) \right]_0^a = -F(0) + F(a) \\ = \int_0^a f(x) \, dx$$

$$(ii) \int_0^\pi x \sin x \, dx = \int_0^\pi (\pi-x) \sin(\pi-x) \, dx \\ = \int_0^\pi (\pi-x) \sin x \, dx \\ = \pi \int_0^\pi \sin x \, dx - \int_0^\pi x \sin x \, dx$$

$$\therefore 2 \int_0^\pi x \sin x \, dx = \pi \int_0^\pi \sin x \, dx \Rightarrow \int_0^\pi x \sin x \, dx = \frac{\pi}{2} \int_0^\pi \sin x \, dx$$

$$(iii) \int_0^\pi x^2 \cos x \, dx = \int_0^\pi x^2 \frac{d \sin x}{dx} \, dx \\ = \left[ x^2 \sin x \right]_0^\pi - \int_0^\pi 2x \sin x \, dx \\ = 0 - \pi \int_0^\pi \sin x \, dx \quad \text{from (i)} \\ = \pi [\cos x]_0^\pi = \pi(-1-1) = -2\pi$$

$$(d) (i) \quad \frac{1}{2} \tan^{-1}(2t+1)$$

$$(ii) \quad t = \tan \frac{\theta}{2}$$

$$: \theta = 0, t = 0$$

$$\frac{dt}{d\theta} = \frac{1}{2} \sec^2 \frac{\theta}{2} = \frac{1}{2} (1+t^2) \quad \theta = \frac{\pi}{2}, t = 1$$

$$\therefore I = \int_0^1 \frac{2 dt}{(1+t^2) \left( \frac{4t}{1+t^2} - \frac{1-t^2}{1+t^2} + 3 \right)}$$

$$= \int_0^1 \frac{2 dt}{4t - 1 + t^2 + 3 + 3t^2}$$

$$= \int_0^1 \frac{2 dt}{4t^2 + 4t + 2}$$

$$= \int_0^1 \frac{2 dt}{(2t+1)^2 + 1}$$

$$= \left[ \tan^{-1}(2t+1) \right]_0^1$$

$$= \tan^{-1} 3 - \tan^{-1} 1$$

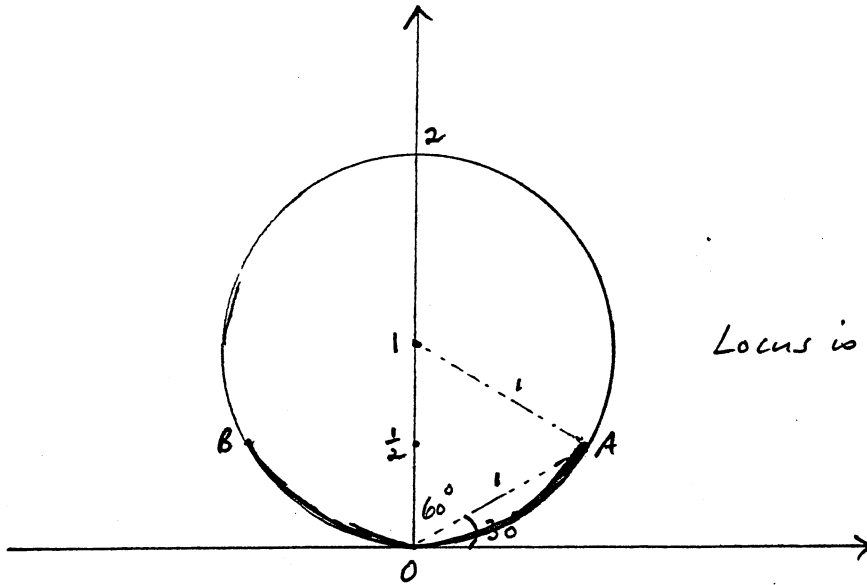
$$= \tan^{-1} \left( \frac{3-1}{1+3 \cdot 1} \right) = \tan^{-1} \frac{2}{4} = \tan^{-1} \frac{1}{2}$$

## Question 2

$$(a) (i) = \sqrt{7+33} = \sqrt{40}$$

$$(ii) x^2 + y^2 = |x+iy|^2 = \frac{|\sqrt{7} + \sqrt{33}i|^2}{|3-i|^2} = \frac{40}{10} = 4$$

(b)



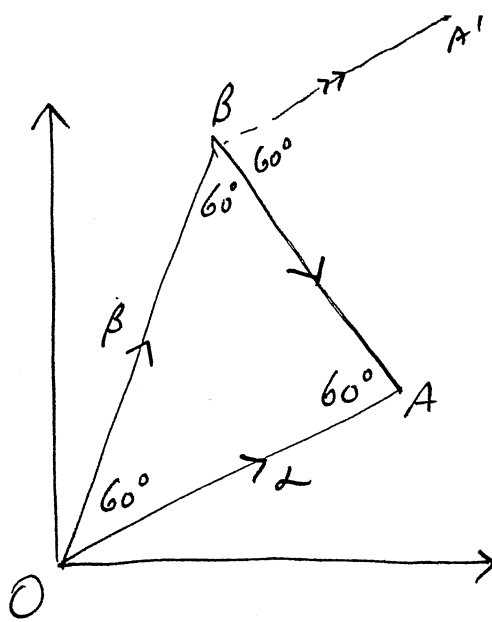
Locus is arc AB,  $y \leq \frac{1}{2}$

$$(c) (i) z = 2 \sin \theta + i (2 \sin \theta \cos \theta) \\ = 2 \sin \theta (\sin \theta + i \cos \theta)$$

$$(ii) z = 2 \sin \theta \left( \cos \left( \frac{\pi}{2} - \theta \right) + i \sin \left( \frac{\pi}{2} - \theta \right) \right)$$

$$\Rightarrow |z| = 2 \sin \theta, \text{ arg } z = \frac{\pi}{2} - \theta \quad \text{since } 0 < \theta < \frac{\pi}{2}$$

(d)



$$(i) \vec{BA} = L - \beta$$

$$(ii) v \vec{BA} = \vec{BA'} = L, \text{ see diagram}$$

$$\text{i.e. } L = v(L - \beta)$$

$$(iii) \text{ Now, } \beta = vL$$

$$\therefore \frac{L}{\beta} = \frac{L - \beta}{L}$$

$$\text{or } L^2 = L\beta - \beta^2$$

$$\text{i.e. } L^2 + \beta^2 = L\beta$$

### Question 3

$$(a) \quad 2(y+1)y' - 2x = 0$$

$$\therefore y' = \frac{x}{y+1}$$

$$\begin{aligned} \therefore y'' &= \frac{y+1 - x y'}{(y+1)^2} = \frac{y+1 - \frac{x^2}{y+1}}{(y+1)^2} \\ &= \frac{(y+1)^2 - x^2}{(y+1)^3} = \frac{1}{(y+1)^3} \end{aligned}$$

$$(b) (i) \quad \Sigma \alpha = 2(\alpha + \beta + \gamma) = 2A \Rightarrow \alpha + \beta + \gamma = A$$

But,  $\alpha + \beta + \gamma$  is a root

$$\therefore A^4 - 2A(A^3) + B = 0 \Rightarrow B = A^4$$

$$(ii) \quad \alpha^2 + \beta^2 + \gamma^2 + (\alpha + \beta + \gamma)^2 = (\Sigma \alpha)^2 - 2\Sigma \alpha\beta$$

$$\Rightarrow \alpha^2 + \beta^2 + \gamma^2 + A^2 = (2A)^2 - 2(0) = 4A^2$$

$$\therefore \alpha^2 + \beta^2 + \gamma^2 = 3A^2$$

$$(c) (i) \quad \frac{u_{k+1}}{u_k} = \frac{\binom{2008}{k} x^k}{\binom{2008}{k-1} x^{k-1}} = \frac{2008! (2009-k)! (k-1)!}{(2008-k)! k! 2008!} \cdot x$$

$$= \frac{2009-k}{k} \cdot x$$

(ii) The middle term is  $u_{1005}$

$$\therefore u_{1005} > u_{1004}$$

$$\text{or } \frac{u_{1005}}{u_{1004}} > 1 \Rightarrow \frac{2009-1004}{1004} x > 1 \text{ from (i)}$$

$$\text{i.e. } x > \frac{1004}{1005}$$

$$\text{Also } u_{1005} > u_{1006}$$

$$\Rightarrow \frac{2009-1005}{1005} x < 1 \text{ from (i)}$$

$$\text{i.e. } x < \frac{1005}{1004}$$

$$\text{i.e. } \frac{1004}{1005} < x < \frac{1005}{1004}$$

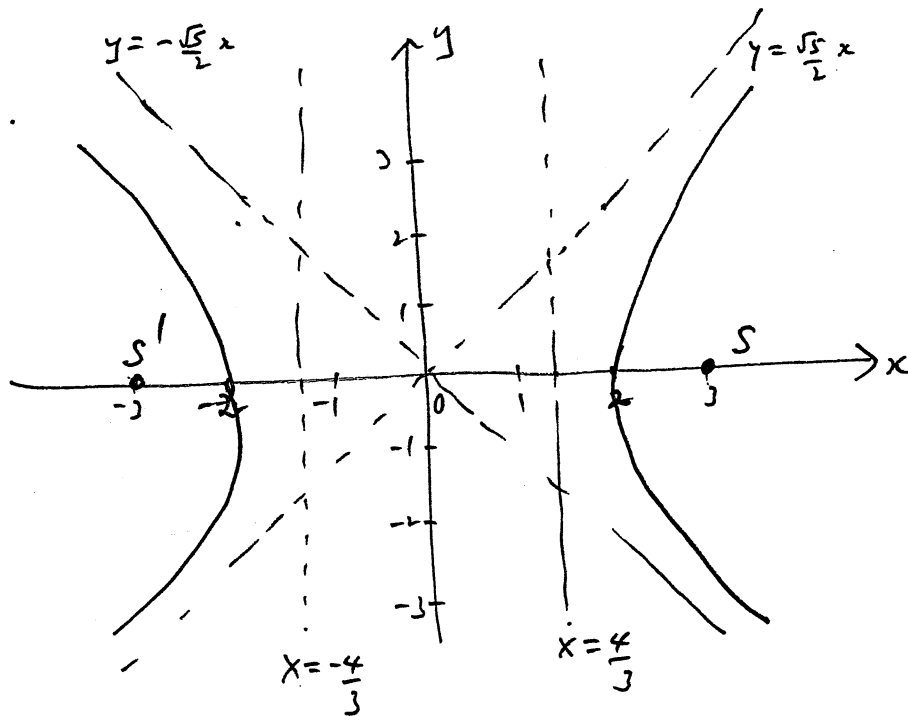
### Question 4

$$(a) (i) c^2 = a^2 + b^2 \Rightarrow c^2 = 4 + 5 = 9, \quad c = 3$$

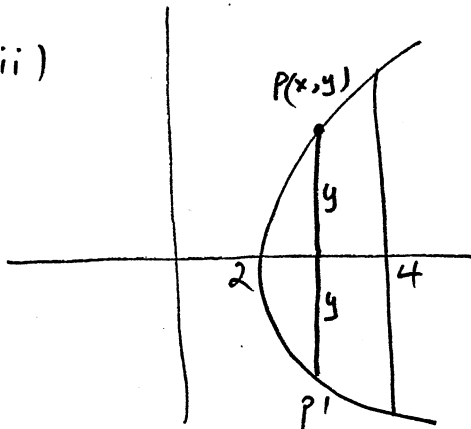
$$e = \frac{3}{2}$$

$\therefore$  foci  $(\pm 3, 0)$ , directrices  $x = \pm \frac{4}{3}$ ,

$$\text{asymptotes } \frac{x}{2} \pm \frac{y}{\sqrt{5}} = 0 \quad \text{i.e. } y = \pm \frac{\sqrt{5}}{2} x$$



(ii)



Take  $P(x, y)$  on curve,  $y \geq 0$

Then, area  $\Delta$  with base  $PP'$

$$= \frac{1}{2} (2y)^2 \sin \frac{\pi}{3} = \sqrt{3} y^2$$

$$\therefore V = \int_2^4 \sqrt{3} y^2 dx$$

$$= \sqrt{3} \cdot 5 \int_2^4 \frac{x^2}{4} - 1 dx$$

$$= 5\sqrt{3} \left[ \frac{x^3}{12} - x \right]_2^4$$

$$= 5\sqrt{3} \left( \frac{16}{3} - 4 - \frac{2}{3} + 2 \right)$$

$$= \frac{40\sqrt{3}}{3}$$

(b) (i) Initially  $x=0$  and  $v>0$

$\Rightarrow$  after  $t=0$  then  $x>0$  i.e.  $\ddot{x}>0$

$\Rightarrow v$  will increase for all  $t$

$$(ii) \frac{d(\frac{1}{2}v^2)}{dx} = x$$

$$\therefore \left[ \frac{1}{2}v^2 \right]_0^v = \left[ \frac{x^2}{2} \right]_0^x$$

$$\text{i.e. } \frac{1}{2}v^2 - \frac{2^2}{2} = \frac{x^2}{2}$$

$$\text{or } v^2 = x^2 + 4$$

$$\therefore v = \sqrt{x^2 + 4} \quad \text{since } v > 0$$

$$(iii) \frac{dx}{dt} = \sqrt{x^2 + 4}$$

$$\therefore \frac{dt}{dx} = \frac{1}{\sqrt{x^2 + 4}}$$

$$t = \int_0^x \frac{1}{\sqrt{x^2 + 4}} dx = \left[ \ln(x + \sqrt{x^2 + 4}) \right]_0^x$$

$$\text{i.e. } t = \ln\left(\frac{x + \sqrt{x^2 + 4}}{2}\right)$$

$$\therefore \frac{x + \sqrt{x^2 + 4}}{2} = e^t$$

$$\text{or } \sqrt{x^2 + 4} = 2e^t - x$$

$$\therefore x^2 + 4 = 4e^{2t} - 4e^t x + x^2$$

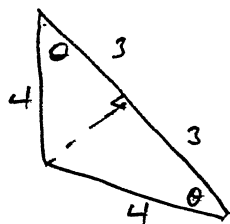
$$\Rightarrow 1 = e^{2t} - e^t x$$

$$\text{or } x = \frac{e^{2t} - 1}{e^t} = e^t - e^{-t}$$



# Question 5

(a) (i)



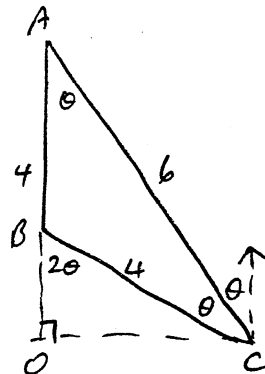
$$\Rightarrow \cos \theta = \frac{3}{4}$$

(ii)  $T_1 \cos \theta + T_2 \cos 2\theta = 40$

$$\therefore T_1 \cos \theta + T_2 (2\cos^2 \theta - 1) = 40$$

$$\Rightarrow \frac{3}{4} T_1 + T_2 \left( \frac{9}{8} - 1 \right) = 40$$

$$\text{i.e. } 6T_1 + T_2 = 320$$



(iii) Resolving in the direction CO,

$$4 \cdot 3 \cdot OC = T_1 \cos\left(\frac{\pi}{2} - \theta\right) + T_2 \cos\left(\frac{\pi}{2} - 2\theta\right),$$

$$\text{where } OC = 4 \sin 2\theta$$

$$\therefore 16 \cdot 9 \cdot \sin 2\theta = T_1 \sin \theta + T_2 \sin 2\theta$$

$$\Rightarrow 16 \cdot 9 \cdot 2 \cos \theta = T_1 + T_2 \cdot 2 \cos \theta$$

$$\text{or } 216 = T_1 + T_2 \cdot \frac{3}{2}$$

$$\therefore \text{From (ii), } 6 \left( 216 - \frac{3T_2}{2} \right) + T_2 = 320$$

$$1296 - 9T_2 + T_2 = 320$$

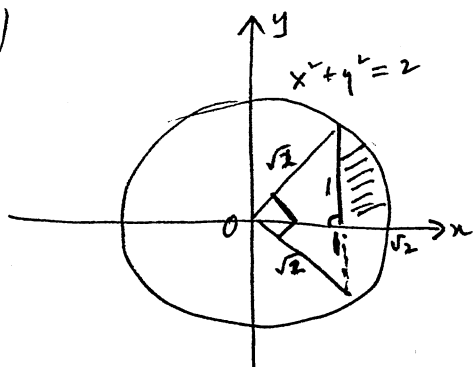
$$\text{or } T_2 = \frac{1296 - 320}{8} = 122 \text{ N}$$

$$\text{or } T_1 = 216 - \frac{3}{2} \times 122 \text{ N} = 33 \text{ N}$$

(b) (i) put  $u = 2 - x^2$  :  $x=1, u=1$   
 $\frac{du}{dx} = -2x$   $x=\sqrt{2}, u=0$

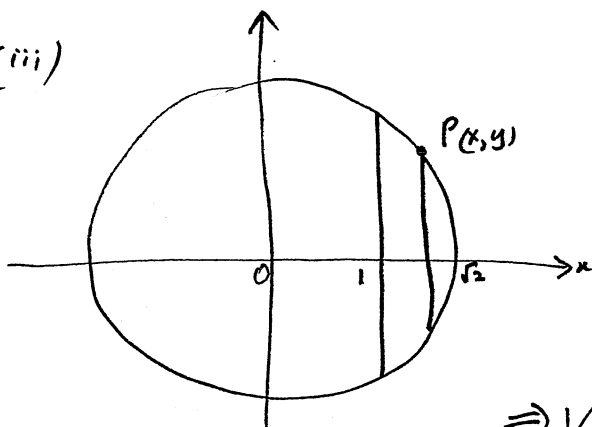
$$\therefore I = \frac{1}{2} \int_0^1 \sqrt{u} du = \frac{1}{3} [u^{3/2}]_0^1 = \frac{1}{3}$$

(ii)



$\therefore$  Shaded area  
 $= \int_1^{\sqrt{2}} \sqrt{2-x^2} dx$   
 $= \frac{1}{2} \left[ \frac{\pi(\sqrt{2})^2}{4} - \frac{1}{2}(\sqrt{2})^2 \right]$   
 $= \frac{\pi}{4} - \frac{1}{2}$

(iii)



Take  $P(x,y)$ ,  $y \geq 0$ , on circle

Then,  $\delta V \approx \pi ((x+dx-1)^2 - (x-1)^2) 2y$

$$\approx 2\pi y \cdot 2(x-1)dx$$

$$\Rightarrow V = 4\pi \int_1^{\sqrt{2}} (x-1)y dx$$

i.e.  $V = 4\pi \int_1^{\sqrt{2}} (x-1)\sqrt{2-x^2} dx$

$$= 4\pi \int_1^{\sqrt{2}} x\sqrt{2-x^2} - \sqrt{2-x^2} dx$$

$$= 4\pi \left( \frac{1}{3} - \left( \frac{\pi}{4} - \frac{1}{2} \right) \right) \text{ from (i) + (ii)}$$

$$= 4\pi \left( \frac{5}{6} - \frac{\pi}{4} \right) = \frac{\pi}{3} (10 - 3\pi)$$

### Question 6

$$(a) \quad (i) \quad m \ddot{x} = -mg - mgk^2 v^2$$

$$\Rightarrow \ddot{x} = -g(1 + k^2 v^2)$$

$$(ii) \quad \ddot{x} = v \frac{dv}{dx} = -g(1 + k^2 v^2)$$

$$\Rightarrow \frac{dv}{dx} = -g \left( \frac{1 + k^2 v^2}{v} \right)$$

$$\therefore -g \frac{dx}{dv} = \frac{v}{1 + k^2 v^2}$$

$$\therefore -g [x]_0^X = \int_V^0 \frac{v}{1 + k^2 v^2} dv$$

$$\begin{aligned} \text{i.e. } -gX &= \frac{1}{2k^2} \left[ \ln(1 + k^2 v^2) \right]_V^0 \\ &= \frac{1}{2k^2} (0 - \ln(1 + k^2 V^2)) \end{aligned}$$

$$\therefore X = \frac{1}{2gk^2} \ln(1 + k^2 V^2)$$

$$(iii) \quad \ddot{x} = \frac{dv}{dt} = -g(1 + k^2 v^2)$$

$$\therefore -g \frac{dt}{dv} = \frac{1}{1 + k^2 v^2}$$

$$\Rightarrow -g [t]_0^T = \frac{1}{k} \left[ \tan^{-1} kv \right]_V^0$$

$$\text{i.e. } -gT = \frac{1}{k} (0 - \tan^{-1} kV)$$

$$\therefore T = \frac{1}{gk} \tan^{-1}(kV)$$

$$(iv) \quad \dot{x} = 0 \Rightarrow x = \frac{V}{g}$$

$$\therefore x_{\max} = -\frac{g}{2} \cdot \frac{V^2}{g^2} + V \cdot \frac{V}{g} = \frac{V^2}{2g}$$

$$\therefore \text{from (ii), } \lim_{k \rightarrow 0} \frac{1}{2gk^2} \ln(1+k^2 V^2) = \frac{V^2}{2g}$$

$$\text{i.e. } \lim_{k \rightarrow 0} \frac{\ln(1+k^2 V^2)}{k^2} = V^2$$

$$(4) (i) \quad |\alpha|^2 + |\beta|^2 - |\alpha - \beta|^2$$

$$= \alpha \bar{\alpha} + \beta \bar{\beta} - (\alpha - \beta)(\overline{\alpha - \beta})$$

$$= \alpha \bar{\alpha} + \beta \bar{\beta} - (\alpha - \beta)(\bar{\alpha} - \bar{\beta})$$

$$= \alpha \bar{\alpha} + \beta \bar{\beta} - (\alpha \bar{\alpha} - \alpha \bar{\beta} - \bar{\alpha} \beta + \beta \bar{\beta})$$

$$= \alpha \bar{\beta} + \bar{\alpha} \beta$$

$$= \alpha \bar{\beta} + \overline{(\alpha \bar{\beta})} = 2 \operatorname{Re}(\alpha \bar{\beta})$$

$$(ii) \quad \cos \theta = \frac{|\alpha|^2 + |\beta|^2 - |\alpha - \beta|^2}{2|\alpha||\beta|} \quad \text{since } \vec{\beta A} = \alpha - \beta$$

$$= \frac{2 \operatorname{Re}(\alpha \bar{\beta})}{2|\alpha||\beta|}$$

$$\Rightarrow |\alpha||\beta| \cos \theta = \operatorname{Re}(\alpha \bar{\beta})$$

## Question 7

(a) (i)  $Q = (a \cos \theta, a \sin \theta)$

(ii)  $\frac{2x}{a^2} + \frac{2y}{b^2} \frac{dy}{dx} = 0 \Rightarrow \frac{dy}{dx} = -\frac{b^2 x}{a^2 y}$

$$= -\frac{b^2}{a^2} \cdot \frac{a \cos \theta}{b \sin \theta} \text{ at } P$$

$$= -\frac{b \cos \theta}{a \sin \theta}$$

$\therefore$  tangent at  $P$  is  $y - b \sin \theta = -\frac{b \cos \theta}{a \sin \theta} (x - a \cos \theta)$

$$\text{or } \frac{\sin \theta}{b} y - \sin^2 \theta = -\frac{\cos \theta}{a} x + \cos^2 \theta$$

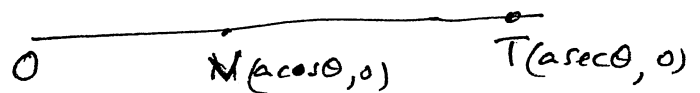
$$\text{i.e. } \frac{\cos \theta}{a} x + \frac{\sin \theta}{b} y = \cos^2 \theta + \sin^2 \theta = 1$$

(iii) (ii)  $\Rightarrow$  tangent at  $Q$  is  $\frac{\cos \theta}{a} x + \frac{\sin \theta}{a} y = 1$  [if  $a=b$ ]

(iv) at  $T$ ,  $\left(\frac{\sin \theta}{b} - \frac{\sin \theta}{a}\right) y = 0 \Rightarrow y = 0$

$$\therefore x = \frac{a}{\cos \theta} = a \sec \theta$$

i.e. we have



$$\therefore ON \cdot OT = |a \cos \theta| |a \sec \theta| = a^2$$

(b) (i) For  $0 < x < \frac{\pi}{2}$ ,  $0 < \cos x < 1$

$$\therefore \cos^n x < \cos^{n-1} x < \cos^{n-2} x$$

$$\Rightarrow u_n < u_{n-1} < u_{n-2}$$

$$(ii) u_n = \int_0^{\frac{\pi}{2}} \cos \theta \cos^{n-1} \theta d\theta$$

$$= \int_0^{\frac{\pi}{2}} \cos^{n-1} \theta \frac{d \sin \theta}{d\theta} d\theta \quad \text{where } u = \cos^{n-1} \theta$$

$$= \left[ \cos^{n-1} \theta \sin \theta \right]_0^{\frac{\pi}{2}} + \int_0^{\frac{\pi}{2}} \frac{du}{d\theta} = -(n-1) \cos^{n-2} \theta \sin \theta$$

$$= \left[ \cos^{n-1} \theta \sin \theta \right]_0^{\frac{\pi}{2}} + \int_0^{\frac{\pi}{2}} (n-1) \cos^{n-2} \theta \sin^2 \theta d\theta$$

$$= 0 + (n-1) \int_0^{\frac{\pi}{2}} \cos^{n-2} \theta (1 - \cos^2 \theta) d\theta$$

$$= (n-1) (u_{n-2} - u_n)$$

$$\therefore u_n (1 + n - 1) = (n-1) u_{n-2} \quad \text{i.e. } u_n = \frac{n-1}{n} u_{n-2}$$

$$(iii) u_n = \left(1 - \frac{1}{n}\right) u_{n-2}$$

$$\therefore \lim_{n \rightarrow \infty} u_n = \lim_{n \rightarrow \infty} u_{n-2} \Rightarrow \text{from (i),}$$

$$\lim_{n \rightarrow \infty} u_n = \lim_{n \rightarrow \infty} u_{n-1}$$

$$(iv) \text{ From (ii), } n u_n u_{n-1} = (n-1) u_{n-1} u_{n-2}$$

$$= (n-2) u_{n-2} u_{n-3}$$

$$= \dots$$

$$= 1 \cdot u_1 u_0$$

$$= \int_0^{\frac{\pi}{2}} \cos \theta d\theta \int_0^{\frac{\pi}{2}} 1 d\theta$$

$$= \left[ \sin \theta \right]_0^{\frac{\pi}{2}} \left[ \theta \right]_0^{\frac{\pi}{2}}$$

$$= 1 \cdot \frac{\pi}{2} = \frac{\pi}{2}$$

(v) From (iv),  $11u_{11}u_{10} = \frac{\pi}{2}$

$$\therefore u_{10} = \frac{\pi}{22} \cdot \frac{693}{256} = \frac{63\pi}{512}$$

(vi) From (iii) and (iv), for large  $n$ ,

$$nu_n u_{n-1} \approx n u_n^2$$

$$\therefore 2008 u_{2008}^2 \approx \frac{\pi}{2}$$

$$\therefore u_{2008} \approx \sqrt{\frac{\pi}{4016}} \quad [\approx 0.028]$$

$$\text{or, of course, } 2009 u_{2008}^2 \approx \frac{\pi}{2}$$

$$\Rightarrow u_{2008} \approx \sqrt{\frac{\pi}{4018}}$$

$$\text{Indeed, } \sqrt{\frac{\pi}{4018}} < u_{2008} < \sqrt{\frac{\pi}{4016}}$$

## Question 8

(a) (i)  $(x+a)(x+b)(x+c)$  and  $(x-a)(x-b)(x-c)$  both have the same leading term  $x^3 \Rightarrow k=1$

$$(ii) \frac{(x+a)(x+b)(x+c)}{(x-b)(x-c)} = x-a + p + \frac{q(x-a)}{x-b} + \frac{t(x-a)}{x-c}$$

$$\text{For } x=a, \frac{2a(a+b)(a+c)}{(a-b)(a-c)} = p$$

$$\therefore q = \frac{2b(b+a)(b+c)}{(b-a)(b-c)}$$

$$t = \frac{2c(c+a)(c+b)}{(c-a)(c-b)}, \text{ on symmetry.}$$

(iii) Put  $x = -a$ , then,

$$0 = 1 - \frac{p}{2a} - \frac{q}{a+b} - \frac{t}{a+c}$$

$$\Rightarrow 0 = 1 - \frac{(a+b)(a+c)}{(a-b)(a-c)} - \frac{2b(b+c)}{(b-a)(b-c)} - \frac{2c(c+b)}{(c-a)(c-b)}$$

$$\therefore \frac{(a+b)(a+c)}{(a-b)(a-c)} + \frac{2b(b+c)}{(b-a)(b-c)} + \frac{2c(c+b)}{(c-a)(c-b)} = 1$$



$$(b) (i) \left(\frac{1}{x}\right)^4 + \left(\frac{1}{x}\right)^3 + 2\left(\frac{1}{x}\right)^2 + 3\left(\frac{1}{x}\right) + 1 = 0$$

$$\text{i.e. } x^4 + 3x^3 + 2x^2 + x + 1 = 0$$

$$(ii) \sum x = -1, \quad \sum \frac{1}{x} = -3$$

$$\therefore \sum \left(x + \frac{1}{x}\right) = -4$$

(iii) The sum of the roots of the quartic is  $-4$   
 $\therefore$  result from (ii)

$$(iv) \alpha, \frac{1}{\alpha}, \beta, \frac{1}{\beta}, \gamma, \frac{1}{\gamma}, \delta, \frac{1}{\delta}$$

$$(v) \left(\alpha + \frac{1}{\alpha}\right)^{-1}, \left(\beta + \frac{1}{\beta}\right)^{-1}, \left(\gamma + \frac{1}{\gamma}\right)^{-1}, \left(\delta + \frac{1}{\delta}\right)^{-1}$$

(vi)  $(x^4 + x^3 + 2x^2 + 3x + 1)(x^4 + 3x^3 + 2x^2 + x + 1) = 0$  has  
 the roots  $\alpha, \frac{1}{\alpha}, \dots, \delta, \frac{1}{\delta}$

$$\equiv (x^2 + 1)^4 + 4x(x^2 + 1)^3 + Ax^2(x^2 + 1)^2 + Bx^3(x^2 + 1) + Cx^4,$$

from (iv)

$\therefore$  Equating coefficients of  $x^5$ ,

$$1 + 2 + 6 + 3 = 12 + B$$

$$\therefore B = 0$$

$$(vii) \text{ From (v), } \sum \left(\alpha + \frac{1}{\alpha}\right)^{-1} = -\frac{B}{C} = 0 \text{ from (vi)}$$

\* Note  $C \neq 0$  since  $\alpha + \frac{1}{\alpha} \neq 0$